

# Belief Merging by Source Reliability Assessment

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## Abstract

Merging beliefs requires the plausibility of the sources of the information to be merged. They are typically assumed equally reliable in lack of hints indicating otherwise [12]; yet, a recent line of research spun from the idea of deriving this information from the revision process itself. In particular, the history of previous revisions [3, 15] and previous merging examples [16] provide information for performing subsequent mergings.

Yet, no examples or previous revisions may be available. In spite of the apparent lack of information, something can still be inferred by a try-and-check approach: a relative reliability ordering is assumed, the merging process is performed based on it, and the result is compared with the original information. The outcome of this check may be incoherent with the initial assumption, like when a completely reliable source is rejected some of the information it provided. In such cases, the reliability ordering assumed in the first place can be excluded from consideration. The first theorem of this article proves that such a scenario is indeed possible. Other results are obtained under various definition of reliability and merging.

## 1 Introduction

Between November 2005 and September 2006 Wikipedia had an article about Porchesia, a 300,000-inhabitant island in the Mediterranean sea [21]. No such place was ever mentioned in the article on the Mediterranean sea or those of Europe, Asia or Africa. It took ten months for the article to be find out an hoax and removed.

Wikipedia accepts information from any user who want to provide some, even anonymously; no prior barrier exists, not even for article creation. This was the reason of its success: a previous project of creating an on-line encyclopedia from reputed experts failed [25]. At the same time, it opens the door to wrong and malicious information. This is not limited to Wikipedia. A number of other successful sites work in a similar way: computer programmers rely on Stack Overflow [2], information on many topics can be found on Stack Exchange [22] and Yahoo Answers [1]. Everyone can provide information; checking is done later: with article removal or correction on Wikipedia, with a measure of trust on other Internet fora.

The Porchesia article is an extreme example, but shows the difficulty of manually checking a large amount of information coming from unknown sources, even by a large and dedicated community [24]. A large island should have been at least mentioned in related articles, such as that about the Mediterranean sea, but it was not. But these articles also come from users of unknown reliability. The conflict is between sources of unknown reliability.

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Unknown does not mean the same for all, as commonly assumed in the theory of belief merging [12, 17, 7, 4, 9, 5]. The user who created the hoax article is not to be trusted much on other additions.

Assuming a prior assessment of the sources would be unrealistic. Not only web sites such as Wikipedia and Stack Overflow rely on new users providing useful information. Even in more controlled scenarios such as database integration assigning priority to sources is only one of several value conflict strategies [18], the others including “ask the user”, “take the most used value”, “if the value is numeric, use the average”. Some works in belief revision try to derive such a preference from the previous history of revisions [3, 15] or from examples [16]. Neither is assumed known in this article.

The scenario considered in this article comprises some sources of information, each providing a number of propositional formulae. In the simplest case of the usual definition of merging, each source gives exactly one formula. More generally, several pieces of information may come from the same source: a sensor produces several readings, a database can be queried several times, etc.

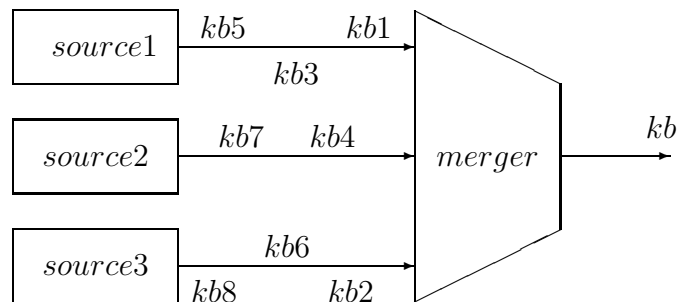


Figure 1: Multiple sources providing formulae to be merged

An example of such settings is in the Figure 1:  $kb_1$ ,  $kb_3$  and  $kb_5$  come from the first source,  $kb_4$  and  $kb_7$  from the second,  $kb_2$ ,  $kb_6$  and  $kb_8$  from the third. The sources may be differently reliable. This affects the result of merging: if the first source is more reliable than the third and  $kb_2$  is the exact opposite of  $kb_1$ , then  $kb_1$  should be accepted and  $kb_2$  rejected. In general, the merger gives preference to formulae coming from reliable sources. Reliability is attached to the sources, and is therefore the same for all formulae provided by the same source. The case of different reliability in the same sources is discussed separately.

The relative reliability of the sources may not be easy to obtain, as witnessed by the large amount of work done assuming equal reliability [12, 17, 7, 4, 9, 5]. A recent line of research has tried to address this problem by deriving this kind of meta-information from the merging process itself: Booth and Meyer and Liberatore [3, 15] used the history of previous revisions to perform the following; Liberatore [16] exploited merging examples.

In lack of an explicit measure of source reliability, a previous history or a set of merging example, the only sensible choice may seem to assume that all formulae are equally reliable. Yet, something more can be said even in this case from the very definition of reliability.

Reliability tells how much each formula affects the result of merging. If the most reliable source says  $x$  and the least says  $\neg x$ , the result should imply  $x$  or at least be consistent with it. Otherwise, merging did not really reflect the considered reliability. In the other way around, the given reliability was not coherent with merging.

This observation can be used as a method for testing a candidate measure of reliability: merging can be attempted using it, and if the result is not coherent with the original assumption then the candidate is discarded. The typical application is with a specific source of

merging that is totally reliable: every piece of information it provides is sure, albeit possibly incomplete; another source providing information that contradicts it cannot be considered reliable. Still better, the result of merging can be compared with the other sources to assess their reliability. This allows for excluding some reliability orderings even in lack of a totally reliable source. This is shown by Theorem 1, which can be depicted as in Figure 2: assuming that  $S_1$  is strictly more reliable than  $S_2$  and  $S_3$ , one possible result of merging is  $\{x, x \wedge \neg y\}$ ; since both  $S_1$  and  $S_3$  are consistent with this set, they are equally reliable (and more so than  $S_2$ ), contrary to the assumption. The initial reliability ordering can therefore be excluded.

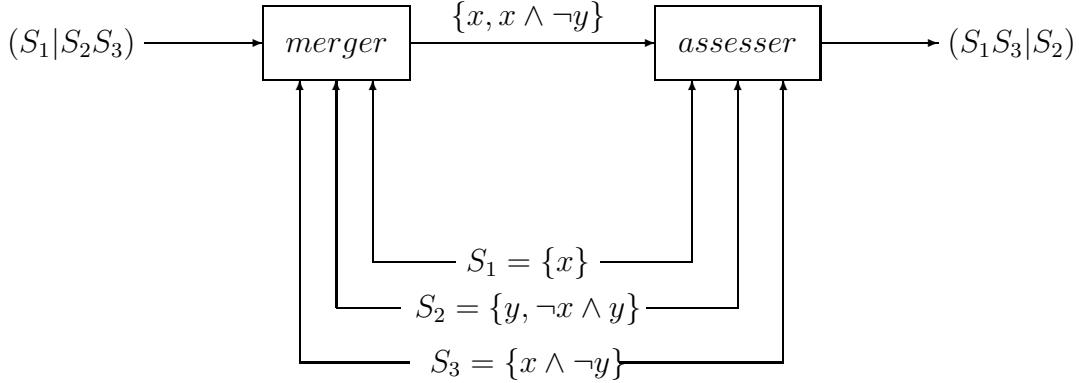


Figure 2: A reliability ordering to exclude

The principle is a fixpoint of the reliability ordering of the sources: if an ordering produces a result that is not coherent with it, it is discarded. An alternative that is considered is that of starting on the assumption of all sources with the same reliability and checking which reliability ordering are obtained after merging.

Some variants are considered (bipartitions with singleton or non-singleton sources, tripartitions, partitions on bounds and weighted merge). The next section provides a roadmap of the results obtained in each case.

## 2 Summary

Section 3 introduces the formalization of the setting: sources are sets of formulae, their relative reliability is formalized as an ordered partitions of these sets. The maxsets of a set and of an ordered partition are defined. Some general results about maxsets follow.

Section 4 contains results about the limit case where every source provides exactly one formula, and sources are partitioned in two classes only: reliable or not. The following results are obtained:

- not all partitions are stable;
- every set of sources has at least one stable partition;
- the maxsets of the stable partitions are the plain maxsets of the set of all formulae.

The last result can be seen as confirmatory: a formula should be more or less taken into consideration depending on how truthful the others provided by the same source are; if every source provides a single formula these formulae should be equally believed, as plain maxsets do.

Section 5 has the simplest case of interest: sources are still partitioned into two classes depending on whether they are reliable or not, but they are not limited to provide a single formula each. A partition may have multiple maxsets, some inducing the partition itself and some others doing not. Such a partition is weakly stable, as opposed to the case of strong stability where every maxset induces the partition itself. The results obtained for this setting are:

- some weakly stable partitions are not strongly stable;
- the maxsets of the weakly stable partitions are the plain maxsets of the set comprising all formulae provided by all sources;
- a weakly stable partition may lead via a maxset to another partition; that partition has a strictly larger first class;
- the maximal weakly stable partitions w.r.t. containment of their first class are the strongly stable partitions;
- the maxsets of the strongly stable partitions are equivalent to the maxsets of a certain partition.

Partitions are related by maxsets: the maxset of a partition induces a possibly different partition. This relation allows for an equivalent definition of strong stability. It is also used in the following section, where it allows for a definition of mild stability when strong stability is unattainable.

Section 6 still maintains sources providing multiple formulae, but partitions them in three classes rather than two: totally reliable, partly reliable and unreliable. These divisions are called tripartitions to distinguish them from the bipartitions of the previous sections; this is necessary because some results relate bipartitions and tripartitions. The following results are obtained:

- a plain maxset is also a maxset of its induced tripartition;
- weakly stable bipartitions correspond to weakly stable tripartitions;
- this is not the case for strong stability;
- the property of containment of the first class of related bipartitions does not extend to tripartitions;
- rather, cycles of related tripartitions are possible;
- some sources do not have strongly stable tripartitions;
- a third kind of stability can be defined as maximality according to the relation among partitions.

Maximal tripartitions may not be strongly stable, as opposed to the case of bipartitions. Defining mild stability as maximality according to the relation between tripartitions, every source has at least a mildly stable partition, even if it has no strongly stable partition.

Section 7 sticks to the principle of assuming a reliability ordering and then matching it with the result of merging, but extends the analysis in several directions:

- divide the sources according to the percentage of correct formulae they provide;
- use an arbitrary order of the sources for expressing reliability;
- merge by weights.

Section 8 explores future directions. One looks particularly applicable in practice: one or more sources are known for certain to be completely reliable. More generally, the reliability ordering may be partially known. Other cases considered in this section are that of unreliable sources considered as intentionally providing false information and non-fixed percentage bounds.

### 3 Sources and maxsets

In this article, a source is identified with the information it provides: a source is a set of formulae. Each such set is assumed consistent. The reliability of the sources is given as an ordered partition of the sources; no other information is assumed known on them.

Every source is assumed to always provide information at the same level of reliability; the case in which a “physical” source may for example produce formulae of two different degrees of reliability may be captured by splitting it in two separate sources.

The example in Figure 1 has three sources. The first provides formulae  $kb_1$ ,  $kb_3$  and  $kb_5$ , and is therefore formalized by the set  $S_1 = \{kb_1, kb_3, kb_5\}$ . The same for the second source  $S_2 = \{kb_4, kb_7\}$  and the third  $S_3 = \{kb_2, kb_6, kb_8\}$ .

The relative reliability of the sources is represented by an ordered partition over the set of sources. Given the three sources  $S_1$ ,  $S_2$  and  $S_3$ , the situation in which the second is more reliable than the other two, which compare the same, is represented by  $(S_2|S_1S_3)$ . More generally, such an ordered partition is denoted by a sequence separated by the symbol  $|$ , where each part is a set of sources; the most reliable ones appear before the first  $|$ , the other follows in order of decreasing reliability.

While  $(S_2|S_1S_3)$  denotes three sources, the similar partition  $(S_2|S_1 \cup S_3)$  represents only two. Yet, the formulae and their reliability are the same: once sources are framed into an ordered partition, merging can be done by first flattening each class into a single set of formulae. This results in an ordered partition of formulae, similar to that used in iterated belief revision [23], which is in fact the same as belief merging from sources of unequal reliability [6].

Ordered partitions of formulae allow for the use of maxsets. The most common definition is from a single-class partition  $(P)$  or, equivalently, the set of formulae  $P$ :

**Definition 1** *A maxset of a set of formulae  $P$  is a consistent subset of  $P$  such that no other consistent subset of  $P$  strictly contains it.*

The word “maxset” is an abbreviation for *maximal consistent subset*. If  $P$  is consistent then its only maxset is  $P$  itself. Otherwise,  $P$  can be made consistent by removing some of its formulae. This can be done in various ways, leading to different maxsets. For example,  $\{x, \neg y, y\}$  can be turned consistent by removing either  $\neg y$  or  $y$ , leading to the two maxsets  $\{x, y\}$  and  $\{x, \neg y\}$ . Maximality ensures that not too much is removed: for example,  $\{x\}$  is also a consistent subset, but deleting both  $y$  and  $\neg y$  for restoring consistency is an overkill.

The maxsets of a single-class partition  $(P)$  are the maxsets of the set  $P$ . A multiple-class partition  $(P_1|P_2|\dots|P_m)$  represents an ordering of preference over formulae in which

$P_1$  ranks first,  $P_2$  second, and so on. If  $P_1 \cup P_2 \cup \dots \cup P_m$  is consistent this ordering does not matter, otherwise it is used to decide which formulae to remove to restore consistency. For example,  $(\neg y|x, y)$  means that  $\neg y$  is to be preferred over both  $x$  and  $y$ ; therefore, the best way to restore consistency is to remove  $y$ , obtaining  $\{\neg y, x\}$ ; the second formula  $x$  is retained, even if it was in the second class of the partition, because it was not involved in the contradiction; in other words, its removal would not help restoring satisfiability. The principle is: remove formulae only if it is necessary to make the set consistent, starting from the first class [19, 20, 23, 6].

**Definition 2** *A maxset of a partition  $(P_1|\dots|P_m)$  is a subset  $M \subseteq P_1 \cup \dots \cup P_m$  such that  $M \cap (P_1 \cup \dots \cup P_i)$  is a maxset of  $(P_1|\dots|P_i)$  for all  $1 \leq i \leq m$ .*

By virtue of the universal quantifier over  $i$ ,  $M \cap P_1$  needs to be a maxset of  $P_1$ . Only when this condition is met the check with  $P_2$  matters. This ensures that formulae of the first class are removed only if this is really necessary to remove contradiction, even disregarding all other formulae. This way, maximality within  $P_1$  takes precedence over maximality within  $P_2$ , which then take precedence over maximality within  $P_3$ , etc.

Collapsing  $(P_1|\dots|P_m)$  into  $P_1 \cup \dots \cup P_m$  makes all formulae compare the same. The maxsets of this set are called the plain maxsets of the partition. Every maxset is also a plain maxset, but not the other way around. Indeed, plain maxsets are obtained by first deleting the ordering over formulae; therefore, some plain maxsets may remove formulae that occur early in the partition and keep others that occur late.

Slightly abusing notation,  $(S_1|S_2S_3)$  means  $(S_1|S_2 \cup S_3)$ . More generally, if two or more sets occur as a class of a partition, that means their union.

Some results carry from maxsets of sets to maxsets of partitions. The first is an almost direct consequence of being a maximal subset: it contains every formula consistent with it.

**Lemma 1** *If  $M$  is a maxset of  $(P_1|\dots|P_m)$  and  $M$  is consistent with  $kb \in P_1 \cup \dots \cup P_m$ , then  $M$  contains  $kb$ .*

*Proof.* By definition,  $M \cap (P_1 \cup \dots \cup P_i)$  is a maxset of  $P_1 \cup \dots \cup P_i$  for every  $i \in \{1, \dots, m\}$ ; for  $i = m$  this condition is that  $M$  is a maxset of  $P_1 \cup \dots \cup P_m$ . If  $kb \notin M$  but  $M \cup \{kb\}$  is consistent, then  $M$  is not a maximal consistent subset of  $P_1 \cup \dots \cup P_m$ , and this condition is part of the definition of  $M$  being a maxset of  $(P_1|\dots|P_m)$ .  $\square$

A maxset of a partition is also a maxset of the set comprising all formulae in the partition. The converse does not hold in general. For example,  $(x|\neg x)$  has a single maxset  $\{x\}$ , but the set  $\{x, \neg x\}$  has the two maxsets  $\{x\}$  and  $\{\neg x\}$ . The second is not a maxset of the partition because it prefers the lower-ranked formula  $\neg x$  over the higher-ranked  $x$ .

If a set is consistent its only maxset is itself. A similar result holds for partitions in different ways: for  $P_1$  only, for  $P_1 \cup \dots \cup P_m$ , or for every union in between.

**Lemma 2** *If  $P_1 \cup \dots \cup P_i$  is consistent, every maxset of  $(P_1|\dots|P_m)$  is a superset of  $P_1 \cup \dots \cup P_i$ .*

*Proof.* By definition of maxsets of a partition,  $M \cap (P_1 \cup \dots \cup P_i)$  is a maxset of  $P_1 \cup \dots \cup P_i$ . Since this union is consistent, its only maxset is itself:  $M \cap (P_1 \cup \dots \cup P_i) = P_1 \cup \dots \cup P_i$ . This implies that  $M$  contains all of  $P_1 \cup \dots \cup P_i$ .  $\square$

The converse holds for partitions made of two classes.

**Lemma 3** *If a maxset of  $P_1 \cup P_2$  is a superset of  $P_1$ , it is a maxset of  $(P_1|P_2)$ .*

*Proof.* Let  $M$  be a maxset of  $P_1 \cup P_2$  such that  $P_1 \subseteq M$ . By definition,  $M$  is a maxset of  $(P_1|P_2)$  if it is a maxset of  $P_1 \cup P_2$  and  $M \cap P_1$  is a maxset of  $P_1$ . The first condition holds by assumption. The second holds as well:  $M$  being a maxset, it is consistent; therefore, its subset  $M \cap P_1$  is consistent; it is maximally consistent within  $P_1$  because  $M \cap P_1 = P_1$ .  $\square$

Given a partition  $(P_1|\dots|P_m)$ , its plain maxsets are obtained by flattening the partition into a set  $P_1 \cup \dots \cup P_m$ . A maxset is also a plain maxset; the converse may hold or not. It holds for two-class partitions under a simple condition.

**Lemma 4** *A plain maxset  $M$  is a maxset of a bipartition  $(P_1|P_2)$  if no other plain maxset  $M'$  is such that  $M \cap P_1 \subset M' \cap P_1$ .*

*Proof.* For two-classes partitions, the definition of maxsets specializes to:  $M \cap P_1$  is a maxset of  $P_1$  and  $M$  is a maxset of  $P_1 \cup P_2$ . The second part is the same as the definition of a plain maxset. The first part is ensured by the statement of the lemma.  $\square$

The case of partitions of two classes is the simplest considered in this article: sources are assessed as either “reliable” or “unreliable”. For example, if the source providing  $S_1$  and  $S_3$  are reliable while  $S_2$  is unreliable, the partition is  $(S_1S_3|S_2)$ . With the abuse of notation announced above, this is the same as  $(S_1 \cup S_3|S_2)$ .

Counterexamples are based on formulae having a given set of maxsets. The following lemma allows avoiding building them explicitly and instead just write for example “formulae  $A$ ,  $B$  and  $C$  such that their maxsets are  $\{A, B\}$  and  $\{B, C\}$ ”.

**Lemma 5 ([16])** *Given some sets of letters, none contained in another, there exists a formula for each letter so that the maxsets of these formulae correspond to the given sets of letters.*

## 4 Two-class partitions of singleton sources

Sources can be either reliable or unreliable. As a further simplification, each may provide a single formula. Both assumptions are released in the following sections; yet, this very simple case already allows for some significant results.

If the classification of sources is given, each maxset of the ordered partition is a way for merging them while retaining as much of them as consistency allows. If the sources  $S_1 = \{x\}$ ,  $S_2 = \{\neg x \wedge \neg y\}$  and  $S_3 = \{\neg x\}$  are partitioned into  $(S_1S_3|S_2)$ , their maxsets are  $\{x\}$  and  $\{\neg x, \neg x \wedge \neg y\}$ . As it is common in belief revision (and in nonmonotonic reasoning in general), when multiple results are possible and nothing suggests that one should be preferred over the others, they are disjointed. In this case, the disjunction of the maxsets reflects the impossibility of choosing one over the other.

When the reliability of the source is not given, the same procedure can be carried on the one-class partition  $(S_1S_2S_3)$ , leading to all plain maxsets of  $S_1 \cup S_2 \cup S_3$ . Alternatively, one may arbitrarily assume a different partition, like  $(S_1S_3|S_2)$ , and then check whether the result of merging is consistent with the assumption. In this case, the maxsets are  $\{x\}$  and  $\{\neg x, \neg x \wedge \neg y\}$ . The first is consistent with  $S_1$ : therefore, the assumption that  $S_1$  is reliable is not contradicted. The same holds for  $S_3$ . However,  $S_2$  also is consistent with the second maxset, contrary to its assumed unreliability.

The following sections use the same principle, which in this simple case can be summarized as:

- assume an arbitrary partition of the sources (reliable and unreliable);
- determine a way to merge them (a maxsets);
- use it to classify the sources: the ones consistent with it are reliable, the others are unreliable;
- check if this classification is the same as that initially assumed.

This section assumes that sources are singletons (each source provides exactly one formula) and that partitions have two classes (reliable and unreliable). The following sections release both constraints, but the general principle stays the same: assume some relative reliability of the sources, merge, use the result for assessing the reliability of the sources; if this is not the same as that originally assumed, it is discarded.

Assuming a partition of the sources, each maxset is a way to merge them. What is missing is the third step of the process: from the result of merging, assess the reliability of the sources. This is done by checking the consistency of the sources with each maxset  $M$ .

**Definition 3** *A set  $M$  induces on the set of sources  $S$  the bipartition  $(R|S \setminus R)$ , where  $R$  contains the sources that are consistent with  $M$ .*

This definition closes the circle: from a partition one can determine its maxsets, and each maxset induces a partition of the source. Technicalities apart, assuming a reliability ordering of the sources allows merging; its result allows assessing the reliability of the sources.

**Definition 4** *A partition is stable if it has a maxset whose induced partition is the partition itself.*

Stability could also be defined by requiring all maxsets to induce the partition itself. This stronger condition is used in the next sections. For this introductory part of the article, the weaker definition is used: if there is a possible way to perform the merge (a maxset) such that the result is consistent with a source, the source is considered reliable. The results obtained from this definition are:

- stable partitions always exist;
- not all partitions are stable;
- the maxsets of the stable partitions are the plain maxsets.

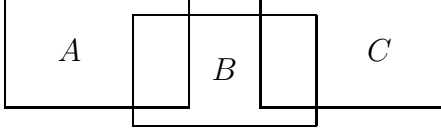
The first two facts show that the definition of stability is not trivial: some reliability orderings lead to merging results that contradict them and some others do not; the constraint of stability gives some useful information in removing some but not all of them from consideration. The third fact may look disappointing: even with the condition of stability still all plain maxsets are obtained. In the next sections the assumption of singleton sources and binary partitions will be lifted, and this result no longer holds. For now, it can be seen as a confirmation that in the basic case the mechanism is sensible: lacking other formulae coming from the same source, the reliability of a formula depends only on itself; therefore, nothing allows assuming a formula more reliable than another.

**Theorem 1** *There exist three formulae  $A$ ,  $B$  and  $C$  such that the partition  $(AB|C)$  is stable while  $(A|BC)$  is not.*



*Proof.* Three consistent formulae  $A$ ,  $B$  and  $C$  are chosen so that their plain maxsets are  $\{A, B\}$  and  $\{B, C\}$ . Lemma 5 proves that this is possible because these sets are not contained in each other; for example,  $A = x \wedge y$ ,  $B = x$  and  $C = x \wedge \neg y$ .

Formulae with such maxsets can be graphically depicted in the space of models, where each formula is a box representing its set of models and intersections indicate mutual consistency.



The partition  $(AB|C)$  is stable. Since  $\{A, B\}$  is consistent but conflicts with  $C$ , it is the only maxset of the partition. The reliability of the sources can be assessed from it. Since  $A$  and  $B$  are consistent with  $\{A, B\}$ , their sources  $\{A\}$  and  $\{B\}$  are in the first class of the partition induced by the maxset. Instead,  $C$  is inconsistent with the maxset. Therefore, its source  $\{C\}$  is in the second class of the induced partition. The resulting partition is therefore  $(AB|C)$ , the same assumed in the first place.

The partition  $(A|BC)$  is not stable. Indeed, the plain maxset  $\{B, C\}$  is excluded because it does not contain the most reliable formula  $A$ . The only plain maxset that is also a maxset of the partition is  $\{A, B\}$ . This set is consistent with  $A$  and  $B$  but not with  $C$ . Therefore, the induced partition is  $(AB|C)$ , while that initially assumed was  $(A|BC)$ .  $\square$

The maxsets of  $\{A, B, C\}$  are  $\{A, B\}$  and  $\{B, C\}$ . The first is generated by the stable partition  $(AB|C)$ ; by symmetry, the second is generated by  $(BC|A)$ . In this particular case, both maxsets are generated by some stable partition. More generally, this happens whenever all sources contain a single formula and partitions are made of two classes.

**Lemma 6** *If  $S$  is a set of singleton sources,  $R \subseteq S$  and  $(R|S \setminus R)$  is a stable partition, then  $R$  is its only maxset.*

*Proof.* By definition of stability,  $(R|S \setminus R)$  has a maxset  $M$  such that  $R$  is exactly the set of elements of  $S$  that are consistent with  $M$ . Since every  $kb \in R$  is consistent with  $M$ , Lemma 1 implies  $kb \in M$ . In other words,  $R \subseteq M$ .

Since  $R$  is the set of elements of  $S$  that are consistent with  $M$ , all other elements of  $S$  are inconsistent with  $M$ , and are therefore not in  $M$ . This implies that  $M = R$ .

Let  $M'$  be an arbitrary maxset of  $(R|S \setminus R)$ . Since  $R = M$  and  $M$  is consistent by definition of maxset, so is  $R$ . By Lemma 2, every maxset of  $(R|S \setminus R)$  is a superset of  $R$ , that is,  $R \subseteq M'$ . But no element of  $S \setminus R$  is consistent with  $M = R$ , implying that none is consistent with  $M'$ . This proves that  $M' = M$ .  $\square$

This lemma is used to prove that the stable partitions generate exactly the plain maxsets of the set of sources if sources are singletons and partitions comprise two classes.

**Theorem 2**  *$R \subseteq S$  is a plain maxset of the set of singleton sources  $S$  if and only if  $(R|S \setminus R)$  is stable.*

*Proof.* By the above lemma, if  $(R|S \setminus R)$  is stable then  $R$  is its only maxset. By definition of maxsets, it is also a maxset of  $R \cup (S \setminus R) = S$ .

Let  $R$  be a plain maxset of  $S$ . For it being a maxset of a partition  $(P_1|P_2)$  it has to be a maxset of  $P_1 \cup P_2$  and  $R \cap P_1$  be a maxset of  $P_1$ . In this particular case,  $P_1 = R$  and  $P_2 = S \setminus R$ .

- The first condition holds because  $P_1 \cup P_2 = R \cup (S \setminus R) = S$ , and  $R$  is a maxset of  $S$  by assumption.
- The second condition holds because  $R \cap P_1 = R \cap R = R$ . This is a maxset of  $P_1$  because  $R$  is consistent (being also a maxset of  $S$ ) and because no other element of  $P_1$  exist at all.  $\square$

The maxsets of the stable partitions can be taken as the result of merging. The above theorem proves that in the case of singleton sources and two-class partitions, this is the same as selecting the plain maxsets of the set of formulae provided by the sources. It also proves that selecting only the maxsets that lead to the same partitions lead to the same result. This was otherwise a possible other definition of the result of merging: instead of selecting all maxsets of all stable partitions, select a maxsets  $M$  only the induced partition  $(R|S \setminus R)$  is stable and in turn generates  $M$ . However, the theorem proves that all maxsets would be obtained.

## 5 Two-class partitions of non-singleton sources

The final theorem of the previous section shows that no selection stems from assuming and then checking the reliability of single-formulae sources: all plain maxsets derive from the stable partitions. This is unsurprising, since the reliability of a formula is the same as the reliability of its source, and could only be altered by the other formulae in its source. This is the case considered in this section: sources with multiple formulae.

**Definition 5** *A set of formulae  $M$  induces on a set of sources  $S$  the partition  $(R|S \setminus R)$  where  $R$  is the set of sources whose formulae are all consistent with  $M$ .*

The partition  $(R|S \setminus R)$  classes the sources as either reliable (those in  $R$ , whose formulae are all consistent with  $M$ ) and unreliable (the others). In this section sources are considered either reliable or not, but sources may also be considered partly reliable. This three-class division is analyzed in the next section. Other variants are discussed at the end of the article.

The partition is on the sources, not on formulae. As a result, even if the same formula is provided by two sources, the first may be in  $R$  but the second does not.

Reliability is assumed and then checked with a fixpoint definition.

**Definition 6** *An ordered partition  $(R|S \setminus R)$  of the sources is **weakly stable** if it has a maxset  $M$  that induces the partition itself.*

This definition requires only that one of the maxsets of the partition to induce the partition itself. But a partition may have multiple maxsets; the others may induce different partitions. Requiring the condition to hold for all of them will be shown to have different consequences.

**Definition 7** *An ordered partition  $(R|S \setminus R)$  of the sources is **strongly stable** if all its maxsets induce the partition itself.*

A different way to restrict the maxsets is to still consider all weak partitions, but only their maxsets that induce the partition itself. It will however been shown that the other maxsets induce weakly stable partitions. In other words, the maxsets that do not induce the partition itself do that for some other weak partition.

Yet another restriction is to start from the partition having all sources in its first class. This realizes the assumption that all sources are initially compared the same. Following the maxsets and the induced partitions may then lead to some weakly or strongly stable partitions. This may seem a restriction since only partitions reachable from the all-equal one are considered. As shown below, it is not.

The results proved in this section for non-singleton sources and reliable–unreliable partitions are:

- restricting to partitions reachable from the all-equal one does not eliminate any weakly or stable partition;
- some weakly stable partitions are not strongly stable;
- a weakly or strongly stable partition may have more than one maxset;
- the maxsets of the weakly stable partitions are the plain maxsets;
- a weakly stable partition may have a maxset that induces a different partition; however, the first class of the latter is larger than that of the former;
- the maximal weakly stable partition w.r.t. containment of their first classes are the strongly stable partitions;
- the maxsets of the strongly stable partitions are the maxsets of the partition  $(\wedge S_1 \dots \wedge S_n | S_1 \cup \dots \cup S_n)$ .

Considering only partitions reachable from the partition that has all sources in its first class is not a restriction at all, even if reachability is limited to a single step.

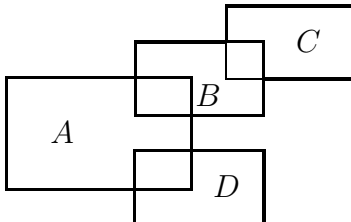
**Theorem 3** *If a partition is weakly or strongly stable then it is induced by a maxset that is also a maxset of the partition where all sources are in the first class.*

*Proof.* Since the partition is stable it has a maxset  $M$  that induces itself. By definition,  $M$  is also a plain maxset. The partition where all sources are in the first class has exactly all plain maxsets, including  $M$ .  $\square$

Every strongly stable partition is also weakly stable, as an immediate consequence of the definition. The converse does not hold: a weakly stable partition may have a maxset that induces a partition that is different from the original one.

**Theorem 4** *There exists a set of sources whose weakly stable partitions are not all strongly stable.*

*Proof.* The sources are  $\{A\}$ ,  $\{B, C\}$  and  $\{D\}$ , the plain maxsets are  $\{A, B\}$ ,  $\{B, C\}$ ,  $\{A, D\}$ . This is a correct description of a merging scenario because sources are identified with the set of formulae they provide, and formulae can be specified by giving their maxsets thanks to Lemma 5. As an example,  $A = x$ ,  $B = y$ ,  $C = \neg x \wedge y$  and  $D = x \wedge \neg y$ . The situation can be depicted in the space of models as follows.



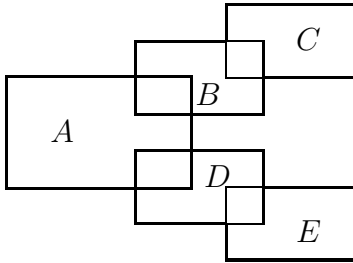
All maxsets of the partition  $(\{A\}|\{B, C\}\{D\})$  contain  $A$  by Lemma 2. Of the formulae of the second class, only  $B$  and  $D$  are consistent with  $A$ , leading to the maxsets  $\{A, B\}$  and  $\{A, D\}$ . Their induced partitions are:

$\{A, B\}$ : the only source that contains only formulae consistent with this maxset is  $\{A\}$ ; as a result, the induced partition is  $(\{A\}|\{B, C\}\{D\})$  itself, showing that this partition is weakly stable;

$\{A, D\}$ : this maxset is consistent with the two sources  $\{A\}$  and  $\{D\}$ ; the induced partition is therefore  $(\{A\}\{D\}|\{B, C\})$ , which is not the same as the partition assumed initially.

The first point shows that  $\{A\}|\{B, C\}\{D\}$  is weakly stable, since it has a maxset that induces the partition itself. The second shows that it is not strongly stable, since the other maxset induces a different partition.  $\square$

A weakly stable partition having a single maxset is strongly stable by definition. However, some strongly stable partitions have more than one maxset, as the following example shows.



The partition  $(\{A\}|\{B, C\}\{D, E\})$  has two maxsets:  $\{A, B\}$  and  $\{A, D\}$ . The source  $\{A\}$  is the only one having only formulae consistent with the first maxset, leading to partition itself. The same for the second maxset. This is a strongly stable partition having two maxsets.

As in the case of singleton sources, the maxsets of the weakly stable partitions are exactly all the plain maxsets of the set of sources. This is proved from a property of the plain maxsets.

**Lemma 7** *If  $M$  is a plain maxset, its induced partition has  $M$  as a maxset.*

*Proof.* Let  $(R|S \setminus R)$  be the partition induced by  $M$ . By definition, all formulae in  $R$  are consistent with  $M$ . If some of these formulae were not in  $M$ , then  $M$  would not be a plain maxset of  $S$ . This proves that  $M$  contains all formulae in  $R$ . By Lemma 3,  $M$  is a maxset of  $(R|S \setminus R)$ .  $\square$

This lemma shows that starting from a plain maxset and moving to its induced partition and then to its maxsets, among them is the original maxset. The partition may also have other maxsets, as shown in the proof of Theorem 4: the maxset  $\{A, B\}$  induces the partition  $(\{A\}|\{B, C\}\{D\})$ , which has two maxsets:  $\{A, B\}$  itself and  $\{A, D\}$ .

This lemma has two side consequences.

The first is related to an additional constraint that was considered above: instead of accepting all maxsets of all weakly stable partition, restrict to those generating the partition. In other words, if a weakly stable partition  $(P_1|P_2)$  has the maxset  $M$ , but this maxset induces a different partition, exclude  $M$  from consideration. This restriction is useless: even if  $M$  induces a different partition  $(P'_1|P'_2)$ , one of the maxsets of this partition is  $M$  itself.

The second consequence is that every set of sources has at least a weakly stable partition. Indeed, every set has at least a maxset  $M$ , and this maxset induces a partition that has  $M$  as a maxset. That partition is therefore weakly stable.

The converse of the lemma holds by definition: the maxsets of a partition are some of the plain maxsets.

**Corollary 1** *The maxsets of weakly stable partitions are the plain maxsets.*

This corollary tells which the maxsets of the weakly stable partitions are: the plain maxsets. However, it is also useful in the other way around. The most obvious way for finding a plain maxset is to start from the empty set and then iteratively adding a formula if consistent with the set. When formulae are provided in a set of sources, this method can be slightly simplified because  $M$  has to be consistent with all formulae in the first class of some partition.

**Algorithm 1 (Maxset and induced partition)**

1.  $M = \emptyset$
2.  $R = \emptyset$
3. for each source  $S_i \in S$ :
  - if  $M \cup S_i$  is consistent:
    - i.  $M = M \cup S_i$
    - ii.  $R = R \cup \{S_i\}$
4. output the weakly stable partition  $(R|S \setminus R)$
5. for each  $S_i \in S \setminus R$ 
  - for each  $F \in S_i$ 
    - if  $S_i \setminus \{F\} \not\subseteq M$  and  $M \cup \{F\}$  is consistent then
  $M = M \cup \{F\}$
6. output maxset  $M$

The additional check  $S_i \setminus \{F_i\} \subseteq M$  allows saving some consistency checks, especially for sources of little cardinality: since  $S_i$  is not in the first class of the partition induced by  $M$ , not all its formulae can be in  $M$ . Therefore, if all of them but  $F$  are in  $M$ , the inconsistency of  $M \cup \{F\}$  is already proved.

This algorithm is guaranteed to find a maxset if all sources are consistent sets, as assumed in this article. However, some maxsets are not found this way, like  $\{x, y\}$  for the sources  $\{\{x, x \wedge \neg y\}, \{y, \neg x \wedge y\}\}$ . Only maxsets of partitions with a non-empty first class are obtained.

Since a partition may have a maxset  $M$  whose induced partition is not itself, the question of which other maxsets that partition has raises. The previous theorem proves only that one of them is  $M$ , but tells nothing about the others. Of particular interest are their induced partitions.

**Theorem 5** *If  $(R|S \setminus R)$  is weakly stable, then all its maxsets include  $\cup R$  and induce a partition  $(R'|S \setminus R')$  such that  $R \subseteq R'$ .*

*Proof.* Since the partition is weakly stable it has a maxset  $M$  that induces the same partition  $(R|S \setminus R)$ . By definition of induced partition, all formulae in  $R$  are consistent with  $M$ . By Lemma 1, since  $M$  is a maxset then it contains them as well. This proves that  $R$  is contained in  $M$ ; therefore,  $R$  is consistent; by Lemma 2, all maxsets of  $(R|S \setminus R)$  include it and all their induced partitions have all of  $R$  in their first class.  $\square$

A partition  $(R|S \setminus R)$  may have a maxset that induces  $(R'|S \setminus R')$  with  $R' \neq R$ , but this implies  $R \subset R'$ . When moving from partition to maxset and then to another partition, either the partitions are the same or the first class is strictly increasing. This forbids the existence of cycles partition—maxset—partition—... unless the partitions are the same. From a different angle, since the first class is strictly increasing, at some point a maximum is hit.

**Theorem 6** *The maximal weakly stable partitions w.r.t. containment of their first class are the strongly stable partition.*

*Proof.* Let  $(R|S \setminus R)$  be a weakly stable partition that is maximal on set-containment of its first class. It is not strongly stable only if one of its maxsets  $M$  induces a partition  $(R'|S \setminus R')$  with  $R \neq R'$ . By Lemma 5,  $R \subset R'$ . Since  $(R'|S \setminus R')$  is induced by  $M$ , it has  $M$  as a maxset by Lemma 7. Therefore, this partition is weakly stable and has its first class strictly containing  $R$ , contrary to the maximality of  $(R|S \setminus R)$ .

The converse is proved by assuming that  $(R|S \setminus R)$  is strongly stable, and proving that it is a maximal weakly stable partition. That it is weakly stable is an immediate consequence of the definitions of stability. Remain to prove that it is maximal. Let  $(R'|S \setminus R')$  be another weakly stable partition with  $R \subset R'$ . By definition of weak stability,  $(R'|S \setminus R')$  has a maxset  $M$  whose induced partition is  $(R'|S \setminus R')$ . By Theorem 5,  $M$  contains  $R'$ , which in turn contains  $R$ . Since every maxset is also a plain maxset, and  $M$  contains  $R$ , by Lemma 3  $M$  is a maxset of  $(R|S \setminus R)$ . As a result,  $M$  is a maxset of  $(R|S \setminus R)$  but its induced partition is  $(R'|S \setminus R')$ , contradicting the assumption of strong stability.  $\square$

This proves the existence of strongly stable partitions for all sources, and also establish a way to find them from weakly stable partitions: following the maxsets and then the induced partitions.

Lemma 7 and Theorem 6 provide a way for finding weakly and strongly stable partitions.

## Algorithm 2 (Weakly and strongly stable partitions of two classes)

1. set  $P$  to be an arbitrary partition, for example  $(S_1 \dots S_n|)$
2. find a maxset  $M$  of  $P$
3. determine the partition  $P'$  that  $M$  induces
4. output  $P'$  as a weakly stable partition (Lemma 7)
5. for each maxset  $M'$  of  $P'$ :
  - if  $M'$  induces a partition  $P'' \neq P'$  then:
    - i. set  $P' = P''$
    - ii. go to Step 5
6. output  $P'$  as strongly stable partition (Theorem 6)

This algorithm works because every maxset  $M$  of a partition  $P$  is also a plain maxset by definition, and the partition induced by  $M$  has  $M$  as a maxset thanks to Lemma 7. Using  $P = (S_1 \dots S_n)$  as the initial partition allows for finding all weakly and stable partitions rather than just one. Indeed, the maxsets of this partitions are exactly the plain maxsets; therefore, all weakly stable partitions can be output in Step 4, and all strongly stable partitions in Step 6 by considering all maxsets instead of just one.

The strongly stable partitions can also be found by a specific partition.

**Theorem 7** *The maxsets of  $(\wedge S_1 \dots \wedge S_n | S_1 \cup \dots \cup S_n)$  are exactly the maxsets of the strongly stable partitions of the sources  $S_1, \dots, S_n$  with the addition to each of every  $\wedge S_i$  consistent with it.*

*Proof.* Let  $S = \{S_1, \dots, S_n\}$  and  $S^\wedge = \{\wedge S_1, \dots, \wedge S_n\}$ . The partition in the statement of the theorem is therefore  $(S^\wedge | \cup S)$ . Let  $a(M)$  be the set obtained by adding to  $M \subseteq S$  all formulae  $\wedge S_i \in S^\wedge$  consistent with it.

Let  $M$  be a maxset of a strongly stable partition  $(R | S \setminus R)$  and prove that  $a(M)$  is a maxset of  $(S^\wedge | \cup S)$ .

If  $M$  is consistent with  $\wedge S_i$ , every formula in  $S_i$  is consistent with  $M$ . Therefore, it is in  $M$  since  $M$  is also a maxset of  $S$ . This implies that all formulae  $\wedge S_i$  consistent with  $M$  are also entailed by  $M$ . Therefore,  $M$  and  $a(M)$  are equivalent.

Since  $M$  is a maxset of  $(R | S \setminus R)$ , it is a maxset of  $S$ . Since  $a(M)$  contains all elements of  $S^\wedge$  that are consistent with  $M$ , it is a maxset of  $S^\wedge \cup (\cup S)$ . In other words, it is a plain maxset of  $(S^\wedge | \cup S)$ . This is the first part of the definition of  $a(M)$  being a maxset of  $(S^\wedge | \cup S)$ ; the other that is still to be proved is that  $a(M) \cap S^\wedge$  is a maxset of  $S^\wedge$ .

The converse is possible if a source is consistent with  $a(M) \cap S^\wedge$  but is not contained in it. For the sake of clarity, the sources can be re-indexed so that  $a(M) \cap S^\wedge = \{\wedge S_1 \dots \wedge S_i\}$  and the source that is consistent but not contained in this set is  $S_{i+1}$ .

Using such indexes, the partition induced by  $M$  is  $(S_1 \dots S_i | S_{i+1} \dots S_n)$ ; this is the same as  $(R | S \setminus R)$  thanks to strong stability. Since  $M$  is a maxset, it is consistent. Therefore, its equivalent set  $a(M)$  and its subset  $a(M) \cap S^\wedge$  are consistent as well. The latter is also consistent with  $\wedge S_{i+1}$  by assumption; therefore, all formulae in  $S_{i+1}$  can be added to it while retaining consistency. Iteratively adding formulae from  $S_{i+2} \dots S_n$  produces a maxset  $M'$  of the partition  $(R | S \setminus R)$ . The partition induced by  $M'$  has  $S_{i+1}$  in the first class and is therefore different than  $(R | S \setminus R)$ , contrary to the assumption of strong stability.

The second part of the proof is to show that every maxset of  $(S^\wedge | \cup S)$  is also a maxset of a strongly stable partition of  $S$  after the formulae  $\wedge S_i$  are removed from it. This is achieved by showing that the partition induced by the maxset is strongly stable.

Let  $M$  be a maxset of  $(S^\wedge | \cup S)$ . Let  $M' = M \cap S^\wedge$  and  $M'' = M \cap \cup S$ . Since  $M$  is maximally consistent,  $\wedge S_i \in M'$ ,  $M'' \cup S_i \not\models \perp$  and  $S_i \subseteq M''$  are equivalent to each other. This proves that  $M$  and  $M''$  are equivalent.

The partition induced by  $M''$  on  $S$  is  $(R | S \setminus R)$ , where  $R = \{S_i \mid M'' \cup S_i \not\models \perp\}$ , which can be rewritten as  $R = \{S_i \mid \wedge S_i \in M'\}$ . Since  $M'$  is consistent, so is  $R$ .

This partition is not strongly stable only if it has a maxset  $M'''$  that induces a different partition. By Lemma 2, since  $R$  is consistent  $M'''$  contains all of it. Therefore, the partition induced by  $M'''$  has all of  $R$  in the first class. This partition can be different than  $(R | S \setminus R)$  only if some  $S_i \in S \setminus R$  is in its first class, which requires it to be consistent with  $M'''$ , and therefore with  $R$ . This implies that the formulae of  $S^\wedge$  that are consistent with  $M'''$  form a strict superset of those consistent with  $M'$ , contrary to the assumption that  $M$  is a maxset.  $\square$

## 6 Unreliable sources

The distinction between reliable and unreliable source may be too crude: among the unreliable sources, some may provide useful information while others do not. This observation leads to partitions of three classes: reliable sources, unreliable sources and source providing only false information.

The same fixpoint definition of the previous sections can be used also in this case: a three-class ordered partition is assumed, and its maxsets are found; each induces a partition, where the first class is defined as before, but the second comprises only the sources that provide some formulae that are consistent with the maxset. The third class contains all other sources, whose formulae are all inconsistent with the maxset.

Weak and strong stability are defined as before: a partition is weakly or strongly stable if one or all of its maxsets induce the partition itself.

The only difference between the previous cases is the definition of induced maxset, which is now composed of three classes. The rest of the framework is as before.

**Definition 8** *A set of formulae  $M$  induces on the set of sources  $S = \{S_1, \dots, S_n\}$  the tripartition  $(R|P|U)$  where*

$$\begin{aligned} R &= \{S_i \mid M \not\models \neg A \text{ for all } A \in S_i\} \\ P &= \{S_i \mid M \not\models \neg A \text{ for some but not all } A \in S_i\} \\ U &= \{S_i \mid M \models \neg A \text{ for all } A \in S_i\} \end{aligned}$$

This definition can be used in the same fixpoint condition of the previous sections, so that a three-class partition is stable if its maxsets induce the partition itself. Depending on whether all or some of these maxsets do, stability is strong or weak. Theorem 3 still holds: considering only partitions that are reachable from the one in which all sources are compared the same is not a restriction. In a further section even more refined classifications of the sources are considered.

The results in this section concern ordered partition of three sources. Some relate partitions of two classes and partitions of three. To avoid confusion, these are called *bipartitions* and *tripartitions*, respectively. Given the number of theorems in this section, a summary is in order:

- the tripartition induced by a plain maxset has it as a maxset;
- weakly stable bipartitions correspond to weakly stable tripartitions and vice versa;
- strongly stable tripartitions may not correspond to strongly stable bipartitions;
- some weakly stable bipartitions can be turned into strongly stable tripartitions, but not all;
- the maxsets of weakly stable tripartitions contain all of  $R$ , like for bipartitions; this however does not extend to  $P$  nor to  $U$ ; it is also not always the case that if  $M$  is a maxset of  $(R|P|U)$  not inducing the same partition, then  $M \cap R$  is strictly larger than  $R$  (this was the case for bipartitions);
- cycles of weakly stable tripartitions are possible; there are even cases where this holds for all maxsets;



- not all sets of sources have strongly stable partitions;
- the relation linking weakly stable partitions and maxsets is not transitive nor symmetric; the latter holds even if the partitions have the same  $R$ ; furthermore, the relation can be made to include an arbitrary graph of partitions.

The first result carries Lemma 7 from bipartitions to tripartitions.

**Theorem 8** *If  $M$  is a plain maxset of  $S$ , it is also a maxset of its induced tripartition.*

*Proof.* Let  $(R|P|U)$  be the partition induced by  $M$ . By definition,  $M$  is a maxset of this tripartition if it is a maxset of  $R \cup P \cup U$ , its intersection with  $R \cup P$  a maxset of  $R \cup P$  and its intersection with  $R$  a maxset of  $R$ . The first holds by assumption, as  $M$  is a plain maxset. For the other two conditions, since  $M$  is consistent only maximal consistency within  $R \cup P$  and  $R$  is left to prove.

Regarding  $R \cup P$ , by definition of induced partition  $U = \{S_i \mid \forall kb \in S_i . M \models \neg kb\}$ . Since  $M$  is consistent, it does not contain any formula in  $U$  and therefore only comprises formulae of the other two classes:  $M \cap \cup(R \cup P) = M$ . If  $M \cap \cup(R \cup P) \subset M' \cap \cup(R \cup P)$  for some  $M'$  then  $M \subset M'$ ; therefore,  $M'$  is inconsistent because  $M$  is a plain maxset of  $S$ .

Regarding  $R$ , by definition of induced partition  $R = \{S_i \mid \forall kb \in S_i . M \not\models \neg kb\}$ . Since  $M$  is a plain maxset, if it is consistent with  $kb$  then it includes it, otherwise it would not be a maximally consistent subset. As a result,  $M$  contains all formulae from the sources in  $R$ . Therefore, it is a maxset of  $\cup R$  simply because it contains the whole of it.  $\square$

Plain maxsets are also maxsets of a partition if they obey a simple condition.

**Lemma 8** *If a plain maxset of  $R \cup P \cup U$  contains all formulae in  $R$  and none in  $U$ , it is a maxset of the partition  $(R|P|U)$ .*

*Proof.* The claim requires  $M \cap R$  to be a maxset of  $R$  and  $M \cap (R \cup P)$  to be a maxset of  $R \cup P$ . For the first condition:  $M \cap R$  is consistent because  $M$  is consistent (being a plain maxset); it is maximal because it contains all of  $R$ .

The second condition also holds. Since  $M \subseteq R \cup P \cup U$  (because  $M$  is a plain maxset of  $R \cup P \cup U$ ) and  $M \cap U = \emptyset$  (by assumption), it holds  $M \subseteq R \cup P$ . Therefore,  $M \cap (R \cup P) = M$ . Since  $M$  is maximally consistent in  $R \cup P \cup U$ , is also maximally consistent in  $R \cup P$ .  $\square$

The converse is not always the case. A maxset may contain formulae in the third class, as for example  $\{x, y, z\}$  is a maxset of  $(\{x\}|\{y\}|\{z\})$ . This may even happen for weakly stable partitions, where at least a maxset has to contain no formula in the third class by definition.

**Lemma 9** *Some weakly stable tripartitions have maxsets that contain formulae in their third class.*

*Proof.* The maxsets of the the partition  $(\{x \vee y\}|\{\neg x \wedge \neg z, \neg y\}|\{z\})$  are  $\{x \vee y, \neg x \wedge \neg z\}$  and  $\{x \vee y, \neg y, z\}$ . The first induces the partition itself, proving that weakly stable. The second contains the formula in the third class.  $\square$

This result relies on the definition of maxset of a partition, and in particular on the possibility for a maxset to include formulae in the last class if they do not contradict the rest of the maxset. Such an event may be an hint that the partition is not sensible, as it classified as unreliable some source providing acceptable information. This motivates the definition of strongly stable partitions, where this cannot happen. Also, a different definition of unreliability is considered in a following section.

As far as weak stability is concerned, the introduction of the third class does not change anything: weakly stable bipartitions and weakly stable tripartitions are the same.

**Theorem 9** *For each weakly stable bipartition  $(R|S \setminus R)$  there exists a weakly stable tripartition  $(R|P|U)$  where  $P \cup U = S \setminus R$ , and vice versa.*

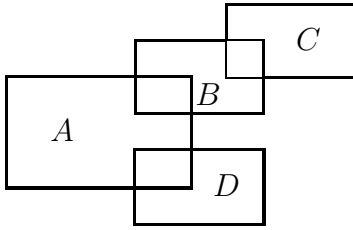
*Proof.* If  $(R|S \setminus R)$  is weakly stable, it has a maxset  $M$  that induces the bipartition itself. The tripartition induced by  $M$  has the same first class of the bipartition because the definition of the first class is the same for bipartitions and tripartitions. Let  $(R|P|U)$  be this tripartition. Since  $M$  is also a plain maxset,  $(R|P|U)$  has  $M$  as a maxset by Lemma 8. Since  $M$  induces  $(R|P|U)$ , this tripartition is weakly stable.

The other direction is proved in the same way. Let  $(R|P|U)$  be a weakly stable tripartition, and  $M$  be a maxset that induces the tripartition itself. The tripartition induced by  $M$  is  $(R|P|U)$ ; since the first class of induced bipartitions and tripartitions are the same, the bipartition induced by  $M$  is  $(R|S \setminus R)$ . By Lemma 7, this bipartition has  $M$  as a maxset, which proves it to be weakly stable.  $\square$

While weakly stable bipartitions and tripartition are the same, apart from the split of the second class, the strongly stable bipartitions and tripartitions exhibit quite a difference. Indeed, a bipartition may not be strongly stable because of a maxset that induces a different bipartition; the addition of the class of unreliable sources may block such a maxset.

**Theorem 10** *There exists a strongly stable tripartition  $(R|P|U)$  such that  $(R|P \cup U)$  is weakly but not strongly stable.*

*Proof.* The claim is proved on the same sources of Example 4:  $S_1 = \{A\}$ ,  $S_2 = \{B, C\}$ , and  $S_3 = \{D\}$ , with maxsets  $\{A, B\}$ ,  $\{B, C\}$  and  $\{A, D\}$ .



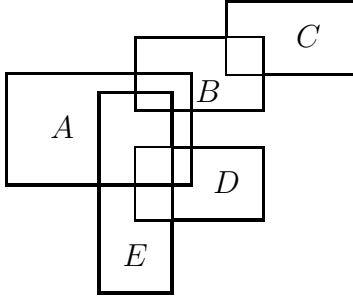
As already proved, the bipartition  $(A|(BC)D)$  is weakly but not strongly stable because of the maxset  $\{A, D\}$ , which induces the different partition  $(AD|(BC))$ . As a result, its other maxset  $\{A, B\}$  is not a maxset of any strongly stable partition, since the partition it induces is  $(A|(BC)D)$ , which is not strongly stable.

The bipartition  $(A|(BC)D)$  is related to the tripartition  $(A|(BC)|D)$  as specified by the claim: the set of all sources and the first class are the same. The tripartition  $(A|(BC)|D)$  is strongly stable, as its only maxset  $\{A, B\}$  induces the tripartition itself: it is consistent with  $A$ , it is consistent with one but not all formulae in  $\{B, C\}$ , and is inconsistent with  $D$ . This proves that a bipartition that is not strongly stable can be turned into a strongly stable tripartition.  $\square$

While some bipartitions can be turned from weakly to strongly stable by splitting their second class, this is not the case for all of them.

**Theorem 11** *There exists a weakly stable tripartition  $(R|P|U)$  such that  $(R|P \cup U)$  is weakly but not strongly stable.*

*Proof.* The sources are  $S = \{A, (BC), (DE)\}$  and the plain maxsets of  $S$  are  $\{A, B, E\}$ ,  $\{B, C\}$  and  $\{A, D, E\}$ .



The bipartition  $(A|(BC)(DE))$  is weakly but not strongly stable. Indeed, it has two maxsets:  $\{A, B, E\}$  and  $\{A, D, E\}$ . The first induces the partition itself, proving it weakly stable. The second induces  $(A(DE)|(BC))$ , proving that stability is not strong.

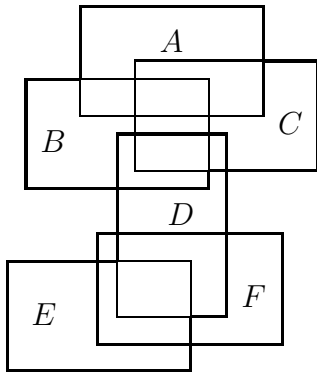
The maxset  $\{A, B, E\}$  induces the tripartition  $(A|(BC)(DE)|)$ . This tripartition has two maxsets:  $\{A, B, E\}$  and  $\{A, D, E\}$ . The second induces the different tripartition  $(A(DE)|| (BC))$ . As a result, the tripartition is weakly but not strongly stable.  $\square$

Finally, some strongly stable bipartition cannot be turned into a strongly stable tripartition, no matter how the second class is split. It will indeed be proved that some sets of sources do not have any strongly stable tripartition, while strongly stable bipartitions are guaranteed to exist as a consequence of Theorem 6 or of Theorem 7. Such a set is provided in the following Corollary 2.

Theorem 5 proves a number of results about the first class of weakly stable bipartitions: they have a consistent first class, their maxsets contain all of it, and the partition they induce have a larger or equal first class. All of this extend to tripartitions because the proof was based on the first class, which is defined in the same way. However, nothing like this holds for the second and the third class of a tripartition.

**Theorem 12** *There exists a tripartition  $(R|P|U)$  such that one of its maxsets induces a partition  $(R'|P'|U')$  such that  $P \not\subseteq P'$ ,  $U \not\subseteq U'$  and  $R \cup P \not\subseteq R' \cup P'$ .*

*Proof.* The claim is proved using the sources  $\{A, B, C\}$ ,  $\{D, E\}$  and  $\{F\}$  with the plain maxsets  $\{A, B, C\}$ ,  $\{B, C, D\}$  and  $\{D, E, F\}$ .



A weakly stable partition of these sources is  $((ABC)(DE)|F)$ , since its maxset  $\{B, C, D\}$  induces the partition itself.

This partition also has the two maxsets  $\{A, B, C\}$  and  $\{D, E, F\}$ . The first maxset  $\{A, B, C\}$  induces  $((ABC)|| (DE)F)$ , which shows that the second class does not necessarily increase, neither does its union with the first. The second maxset  $\{D, E, F\}$  induces  $((DE)F|| (ABC))$ , whose third class is incomparable with that of the original partition.  $\square$

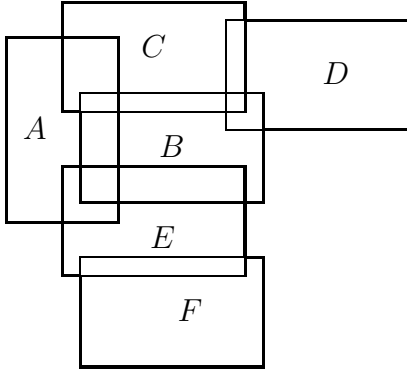
Incidentally, the set of sources has a strongly stable tripartition:  $((ABC)|| (DE)F)$ . Indeed, the only maxset of this tripartition is  $\{A, B, C\}$ , which induces the partition itself.

In this and in the previous proof, going from a partition to a maxset and to its induced partition reduces the second class. This is however easy to show not to always be the case, with the same sources but  $\{F, G\}$  in place of  $\{F\}$ , where  $G$  is only consistent with  $F$ . Thanks to this change, the maxset  $\{D, E, F\}$  induces the partition  $((DE)|(FG)|(ABC))$ .

Another common feature of the above proofs is that the tripartitions that are weakly but not strongly stable have a maxset whose induced partition has a second class that is strictly larger than the original partition. This is however not always the case.

**Theorem 13** *There exists a partition with two maxsets such that the first induces the partition itself and the second induces a partition with a smaller second class.*

*Proof.* The proof is based on the partition  $(A|(BCD)(EF)|)$  whose plain maxsets are  $\{A, B, C\}$ ,  $\{A, B, E\}$ ,  $\{B, C, D\}$  and  $\{E, F\}$ .



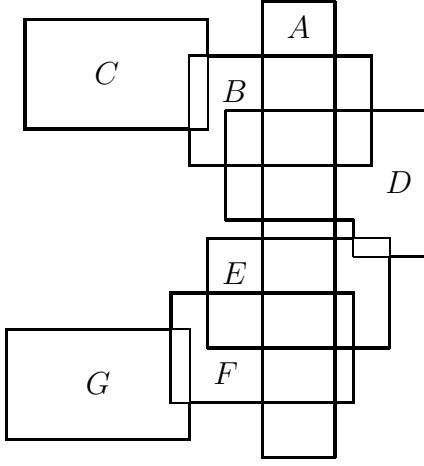
One of the maxsets of the tripartition  $(A|(BCD)(EF)|)$  is  $\{A, B, E\}$ , which induces the partition itself. This is the first part of the claim. The only other maxset of the tripartition is  $\{A, B, C\}$ , which induces  $(A|(BCD)|(EF))$ . This partition has the same first class of the original one, but a smaller second class.  $\square$

**Theorem 14** *There exists a partition with two maxsets such that the first induces the partition itself and the second induces a partition with a smaller second class.*

For bipartitions, a strongly stable partition can always be found by starting from a maxset and then following the induced partition and then its maxsets, or starting from a weakly stable partition and following a similar path. Nothing like this can be done on tripartitions: one may start from a weakly stable one, follow the maxsets and then the induced partitions and so only only to end up in the original tripartition. This can be the case even for all maxsets of a tripartition, as the following example shows.

**Lemma 10** *There exists two tripartitions  $T$  and  $T'$  that have the same first class and the same two maxsets  $M$  and  $M'$ , where  $M$  induces  $T$  and  $M'$  induces  $T'$ .*

*Proof.* The sources are  $\{A\}$ ,  $\{B, C\}$ ,  $\{D, E\}$ ,  $\{F, G\}$  and their plain maxsets are  $\{B, C\}$ ,  $\{A, B, D\}$ ,  $\{D, E\}$ ,  $\{A, E, F\}$  and  $\{F, G\}$ .



The tripartitions and the maxsets of the claim are:

$$\begin{aligned}
T &= (A|(BC)(DE)|(FG)) \\
T' &= (A|(FG)(DE)|(BC)) \\
M &= \{A, B, D\} \\
M' &= \{A, E, F\}
\end{aligned}$$

Both tripartitions have  $A$  only in their first class. All their maxsets therefore contain  $A$ . The only plain maxsets containing  $A$  are  $M$  and  $M'$ . Both are maxsets of the tripartitions, since they cannot be consistently added a formula of the second class even removing the formulae of the third ( $F$  for the first tripartition and  $B$  for the second). That  $M$  induces  $T$  and  $M'$  induces  $T'$  is a consequence of the definition.  $\square$

A first consequence of this lemma is the existence of strongly stable bipartitions that cannot be turned into strongly stable tripartitions by splitting their second class.

**Corollary 2** *There exists a strongly stable bipartition  $(R|S \setminus R)$  such that  $(R|P|U)$  is not strongly stable for any  $P$  and  $U$  such that  $R \cup P \cup U = S$ .*

*Proof.* The tripartitions  $T$  and  $T'$  of the previous theorem are weakly but not strongly stable tripartition. Since their first classes coincide, they are associated the same bipartition, which is strongly stable because its maxsets  $M$  and  $M'$  induce the tripartition, and therefore induce the same bipartition.  $\square$

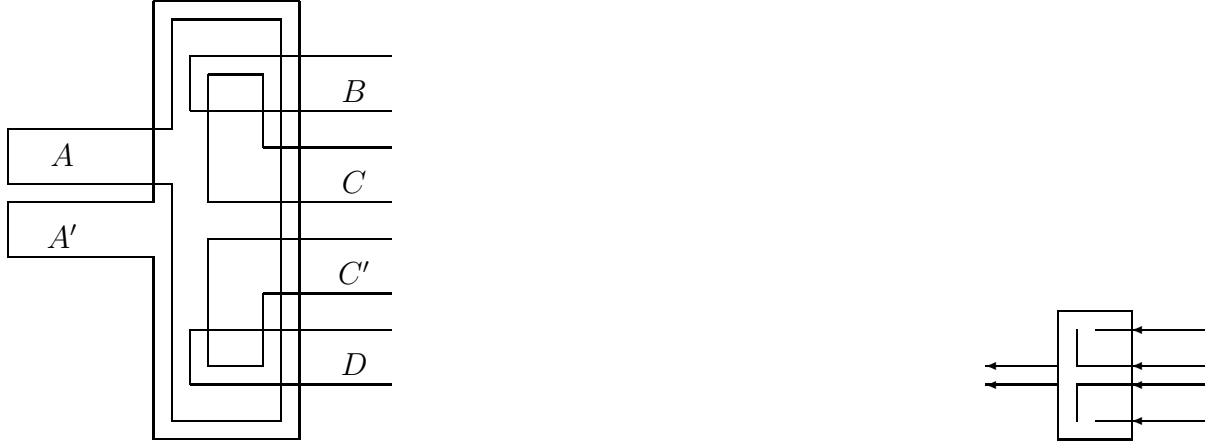
Lemma 10 also proves that a strongly stable tripartition cannot be found by moving from tripartitions to maxsets and then to their induced tripartitions. A strongly stable tripartition requires such a path being a loop of length two (tripartition – maxset – same tripartition), while the lemma shows a loop of length four.

Incidentally, the sources of the proof have a strongly stable tripartition:  $((BC)||A(DE)(FG))$ ; its only maxset is  $\{B, C\}$ , which induces the partition itself. This means that the tripartition – maxset – tripartition procedure may fail to find a strongly stable tripartition even if one exists. However, it is also the case that a set of sources has no strongly stable tripartition.

**Theorem 15** *Some sets of sources have no strongly stable tripartitions.*

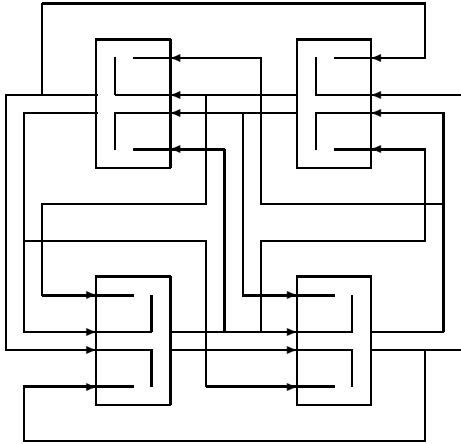
*Proof.* The claim is proved using four sources:  $\{A, A'\}$ ,  $\{B, B'\}$ ,  $\{C, C'\}$ , and  $\{D, D'\}$ . The maxsets containing both  $A$  and  $A'$  are  $\{A, A', B, C\}$  and  $\{A, A', C', D\}$ ; a similar pair of maxsets contain both  $B$  and  $B'$ , another  $C$  and  $C'$  and yet another  $D$  and  $D'$ .

Graphically, these formulae can be seen as a composition of four modules, one for each source. The module for the first source is as follows, with the smaller figure on the right being a simplified representation.



The modules are composed so that formulae do not intersect outside them. There are two maxsets for each source. For the first, they are  $\{A, A', B, C\}$  and  $\{A, A', C', D\}$ . They induce the partitions  $((AA')|(BB')(CC')|(DD'))$  and  $((AA'|(CC')(DD'))|)$ , respectively; both of them have both maxsets. As a result, they are not strongly stable.

The claim requires that a set of sources can be build in a symmetric way, so that what holds for a source holds for the others as well. The four modules, one for each source, are deployed in circle:



The first module has  $A$  and  $A'$  as “output” and  $B, C, C', D$  as “input”; the second module has  $B$  and  $B'$  as output and  $C, D, D', A$  as input, etc. By symmetry, the argument that holds for the maxsets  $\{A, A', B, C\}$  and  $\{A, A', C', D\}$  carries to all other ones, showing that none of them induces a tripartition that is strongly stable.  $\square$

Given that strongly stable tripartitions do not always exist, a milder condition can be defined. The proof of the theorem is based on four tripartitions such that each induces not only itself but also one of the others, forming a sort of cycle. While none induces only itself, one may consider the group itself to be strongly stable, in the sense that the removal of any element from it falsifies this condition.

Given two partitions  $X$  and  $Y$ , that  $X$  has a maxset that induces  $Y$  is denoted by  $X \Rightarrow Y$ . This relation among partitions allows rewriting the definition of weakly and stably strong

partitions as follows:  $X \Rightarrow X$  for weakly stable and  $X \Rightarrow Y$  implies  $Y = X$  for strongly stable. The difference between bipartitions and tripartitions only affects the definition of induced partition, which is part of the definition of  $\Rightarrow$ .

**Definition 9** *The relation  $\Rightarrow$  is defined as: given two partitions  $X$  and  $Y$ , it holds  $X \Rightarrow Y$  if  $X$  has a maxset that induces  $Y$ .*

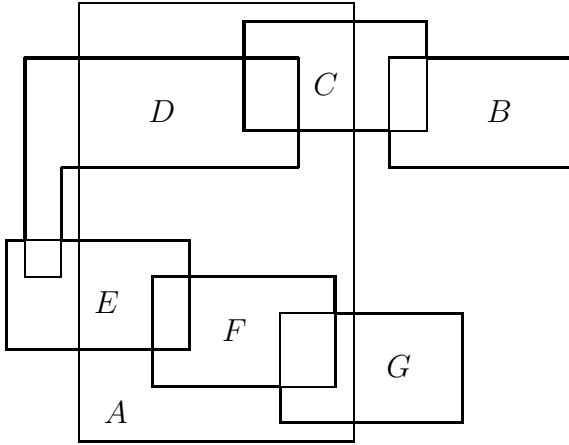
The relation  $\Rightarrow$  is defined over all partitions, including the ones that are not weakly stable. For these,  $X \Rightarrow X$  may not hold, meaning that  $\Rightarrow$  is not reflexive. Restricting to the weakly stable partitions makes  $\Rightarrow$  reflexive, but neither symmetric nor transitive, as the following theorem proves. The relation is however serial: for every tripartition  $X$  it holds  $X \Rightarrow Y$  for some possibly different tripartition  $Y$ .

**Theorem 16** *The relation  $\Rightarrow$  is serial but is not reflexive, not symmetric and not transitive. When restricted to weakly stable tripartitions is reflexive but neither symmetric nor transitive.*

*Proof.* The relation  $\Rightarrow$  is serial if  $X \Rightarrow Y$  holds for every  $X$  and some  $Y$ . This is the case because every tripartition  $X$  has at least a maxset  $M$ , and every maxset like  $M$  induces a tripartition  $Y$ ; therefore,  $X \Rightarrow Y$ .

Not being reflexive is shown by the partition  $(||A)$ : its only maxset is  $\{A\}$ , which induces  $(A||)$ . Restricting to weakly stable tripartitions makes it reflexive by definition, since weak stability is defined as  $X \Rightarrow X$ .

The relation  $\Rightarrow$  is shown not symmetric and not transitive by the following example.



The sources are  $\{A\}$ ,  $\{B, C\}$ ,  $\{D, E\}$  and  $\{F, G\}$ , the plain maxsets  $\{A, C, D\}$ ,  $\{A, E, F\}$ ,  $\{A, F, G\}$ ,  $\{B, C\}$  and  $\{D, E\}$ . The following figure employs arrows to indicate both that a partition has a maxset and that a maxset induces a partition.

$$\begin{array}{ccc}
 (A|(BC)(DE)|(FG)) & \leftrightarrow & ACD \\
 \downarrow & & \uparrow \\
 AEF & \leftrightarrow & (A|(FG)(DE)|(BC)) \\
 & & \downarrow \\
 & & AFG \\
 & & \updownarrow \\
 & & (A(FG)|| (DE)(BC))
 \end{array}$$

Since  $A$  is consistent, the tripartitions  $(A|(BC)(DE)|(FG))$  and  $(A|(FG)(DE)|(BC))$  have only maxsets containing  $A$ . In particular, both have  $\{A, C, D\}$  and  $\{A, E, F\}$ , but only the second has  $\{A, F, G\}$ . This maxset induces the partition  $(A(FD)|| (DE)(BC))$ . This partition has  $\{A, F, G\}$  as its only maxset, showing it to be strongly stable.

In terms of the relation,  $(A|(FG)(DE)|(BC)) \Rightarrow (A(FD)|| (DE)(BC))$  but not the other way around, proving that  $\Rightarrow$  is not symmetric even on weakly stable partitions. Furthermore, both  $(A|(BC)(DE)|(FG)) \Rightarrow (A|(FG)(DE)|(BC))$  and  $(A|(FG)(DE)|(BC)) \Rightarrow (A(FD)|| (DE)(BC))$  hold, yet  $(A|(BC)(DE)|(FG)) \Rightarrow (A(FD)|| (DE)(BC))$  does not hold, proving that the relation is not transitive.  $\square$

The definitions of weak and strong stability can be expressed in terms of the relation  $\Rightarrow$  between tripartitions. Indeed, weak stability requires the partition to have a maxset that induces the partition itself ( $X \Rightarrow X$ ) while strong stability also requires that no other tripartition is obtained this way:

**weakly stable:**  $X \Rightarrow X$

**strongly stable:**  $X \Rightarrow Y$  if and only if  $X = Y$

Weakly stable tripartitions always exist, but some set of sources do not have any strongly stable tripartition. This situation can be avoided by relaxing the second definition. This can be done in at least three different ways, where  $\stackrel{*}{\Rightarrow}$  is the transitive closure of  $\Rightarrow$ , expressing the existence of a path of tripartitions.

**Definition 10** *A tripartition is mildly stable if it is weakly stable and one of the following alternative conditions hold:*

1. if  $X \Rightarrow Y$  then  $Y \Rightarrow X$
2. if  $X \Rightarrow Y$  then  $Y \stackrel{*}{\Rightarrow} X$
3. if  $X \stackrel{*}{\Rightarrow} Y$  then  $Y \stackrel{*}{\Rightarrow} X$

Rather than insisting on  $Y = X$ , these conditions allow for  $X$  to lead to some other tripartition  $Y$ , but only if coming back to  $X$  is possible in some way. In particular, the first requires that a single step suffices, the second allows for a path, and the third disallows for a tripartition to be reachable without the original tripartition being reachable from it.

None of the three conditions imply weak stability, since they only constraint the case where  $X \Rightarrow Y$  without saying anything about  $X \Rightarrow X$ . This is why the definition of mild stability also requires the tripartition to be weakly stable.

Conversely, weak stability does not imply any of them:  $X \Rightarrow X$  only implies that a maxset  $M$  of  $X$  induces  $X$  itself, but  $X$  may also have another maxset  $M'$  that induces a different partition  $Y$ . In this case,  $X \Rightarrow Y$  and  $X \stackrel{*}{\Rightarrow} Y$ , which is the situation the three conditions constraint.

All of them condition the case where  $X \Rightarrow Y$ ; this formula means that starting from the tripartition  $X$  one of the alternative results of merging (one of the maxsets) leads to the partition  $Y$ . All three conditions accept  $X$  if  $X$  and  $Y$  can be somehow be considered “peer alternatives”: given that  $Y$  is an alternative to  $X$ , then  $X$  is an alternative to  $Y$ . If the second does not hold, then  $Y$  is an way “more stable” than  $X$ , since the latter leads to the former but not the other way around.

The second and the third definitions may look the same, but they are not; they are all different. This will be shown after a general lemma is proved on how to build counterexamples.



**Theorem 17** *Some sets of sources have no mildly stable partitions according to the first definition.*

*Proof.* The counterexample is the following: the sources are  $\{A, A', A''\}$ ,  $\{B, B', B''\}$ ,  $\{C, C', C''\}$  and  $\{D, D', D''\}$ . For each source, a group of four maxsets contains it all; for the first source, these are  $\{A, A', A'', B, B', C\}$ ,  $\{A, A', A'', C, C', D\}$  and  $\{A, A', A'', D, D', B\}$ . Another similar group of four maxsets contains all of  $\{B, B', B''\}$ , etc. These maxsets do not contain each other since: a. each has a whole source and a part of the others; and b. the ones that contain the same whole source (like two that contain all of  $\{A, A', A''\}$ ) have the first two formulae of another source (like  $\{B, B'\}$ ) and one of yet another (like  $\{C\}$ ). Since these sets of letters do not contain each other, they can be realized by maxsets thanks to Lemma 5.

Since mildly stable tripartitions are also weakly stable by definition, only tripartitions induced by one of these maxsets can be mildly stable. The first maxset  $\{A, A', A'', B, B', C\}$  induces the following tripartition:

$$((AA'A'')|(BB'B'')(CC'C'')|(DD'D''))$$

This tripartition has two maxsets:  $\{A, A', A'', B, B', C\}$  and  $\{A, A', A'', C, C', D\}$ . The first is a maxset because it induces the partition itself; the second is an alternative because it contains  $C'$  from the second class, which the first does not. This maxset induces a different partition:

$$((AA'A'')|(CC'C'')(DD'D'')|(BB'B''))$$

Again, this tripartition has the maxset  $\{A, A', A'', C, C', D\}$  because it was induced by it, but also  $\{A, A', A'', D, D', B\}$  because this maxset contains  $D'$  from the second class, which is not in the previous maxset. This maxset induces the partition:

$$((AA'A'')|(DD'D'')(BB'B'')|(CC'C''))$$

This tripartition has the maxset  $\{A, A', A'', D, D', B\}$  but also  $\{A, A', A'', B, B', C\}$  because the latter has  $B'$  while the first does not. This was the maxset the proof started from. Therefore, a cycle of three tripartition has been shown:

$$\begin{array}{ccc}
& ((AA'A'')|(BB'B'')(CC'C'')|(DD'D'')) & \\
& \swarrow & \uparrow \\
((AA'A'')|(CC'C'')(DD'D'')|(BB'B'')) & & \\
& \searrow & \\
& ((AA'A'')|(DD'D'')(BB'B'')|(CC'C'')) &
\end{array}$$

The point of the proof is that, apart from  $\{A, A', A''\}$ , each maxset has two formulae in a source  $S$  and one in another  $S'$ ; its induced tripartition has  $S$  and  $S'$  in the second class. However, another maxset has two formulae in  $S'$ , and is therefore an alternative maxset of the same tripartition. Its induces tripartition has  $S$  in the third class and another source in the second. A mechanism like this allows building cycles of tripartitions and maxsets of arbitrary length.

What holds for the maxsets containing  $\{A, A', A''\}$  holds for the maxsets containing  $\{B, B', B''\}$ , for the maxsets containing  $\{C, C', C''\}$  and for the maxsets containing  $\{D, D', D''\}$  because these are the same apart from a renaming of the formulae. Therefore, this set of sources has no mildly stable partition according to the first definition.  $\square$

The mechanism used in the proof can be generalized so that counterexamples and specific scenarios can be easily created. This is in the same line of Lemma 5, which shows that sets of letters always express maxsets given that no set contain another.

For tripartitions, the construction is not that clean. Ideally, it should be possible to give an arbitrary graph and obtain a set of sources so that their tripartitions are related as in the graph. Instead, the following lemma shows how to obtain a graph only for the weakly stable tripartitions, and only if the graph has a specific form.

**Lemma 11** *Given a directed graph of  $n$  disconnected subgraphs of  $n$  nodes each, a set of sources exists such that the graph expresses its tripartition-maxset-tripartition relation.*

*Proof.* Let  $X_j^i$  be the  $j$ -th node of the  $i$ -th subgraph. The existence of an edge from  $X_j^i$  to  $X_z^i$  is denoted by  $X_j^i \rightarrow X_z^i$ .

The sources are  $S_0, S_1, \dots, S_n$  where  $S_i = \{F_i, F'_i, F''_i\}$ . The first subgraph is obtained by the following maxsets:

$$M_j^0 = \{F_0, F'_0, F''_0, F_j, F'_j\} \cup \{F_z \mid X_j^i \rightarrow X_z^i\} \text{ for every } j \in [1, n]$$

The maxsets for the other subgraphs are described below, for the moment the only important part of the construction is that none of them contain all three formulae  $F_0$ ,  $F'_0$  and  $F''_0$ .

The weakly stable tripartitions are those induced by at least a maxset. The tripartition  $P_j$  induced by  $M_j^0$  has  $S_0 = \{F_0, F'_0, F''_0\}$  in the first class,  $S_j = \{F_j, F'_j, F''_j\}$  in the second together with all  $S_z = \{F_z, F'_z, F''_z\}$  such that  $X_j^0 \rightarrow X_z^0$ . The remaining sources are in its third class.

$$P_j^0 = (S_0 | S_j S_z \dots | \dots)$$

Since  $S_0$  is consistent, only plain maxsets containing all of  $S_0$  can be maxsets of this tripartition  $P_j^0$ . These are:  $M_j^0$ , the maxsets  $M_z^0$  such that  $X_j^0 \rightarrow X_z^0$  contain all formulae in the first class, and the other maxsets  $M_w^0$ . These three kinds of maxsets are considered in turn:

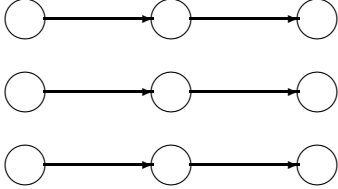
- $M_j^0$  is a plain maxset and it contains all formulae in the first class and none in the third; by Lemma 8, it is a maxset;
- let  $z$  be such that  $X_j^0 \rightarrow X_z^0$ ; by construction,  $M_z^0$  contains all formulae in the first class and  $F'_z$ ; no other maxset contains all these four formulae; therefore, this one is maximal both in the first class and in the union of the first and the second;
- the other plain maxsets  $M_w^0$  are the only ones containing all formulae in the first class of the tripartition; of the second class they may contain  $F_j$ , but neither  $F'_j$  (because only  $M_j$  does) nor  $F''_z$  (only  $M_z^0$  does); the intersection of this maxset with the first and the second class of the partition is therefore strictly contained in  $M_j^0$ .

These three points prove that the maxsets of the tripartition  $P_j^0$  are  $M_j^0$  and all  $M_z^0$  with  $X_j^i \rightarrow X_z^i$ . By construction,  $M_j^0$  induces  $P_j^0$  while  $M_z^0$  induces  $P_z^0$ . Therefore, every  $P_j^0$  is a weakly stable partition, and  $P_j^0 \Rightarrow P_z^0$  if and only if  $X_j^0 \rightarrow X_z^0$ .

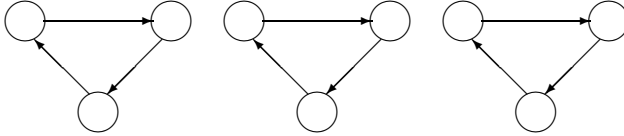
By symmetry, what holds for index  $i = 0$  holds for the others. For example, the maxset  $M_j^1$  induces  $P_j^1$  for every  $j \neq i$ . What makes all these cases disconnected is that  $M_j^i$  contains all three formulae  $F_i, F'_i, F''_i$ , while no other  $M_j^w$  contains  $F''_i$ .

The only caveat to this construction is that for  $i = 0$  the three formulae  $F_0, F'_0, F''_0$  are in all maxsets and are the first class of every partition. For  $i = 1$ , the same role is taken by  $F_1, F'_1, F''_1$ . Therefore, the node  $X_1^1$  cannot be represented by  $P_1^1$  and  $M_1^1$ , but by  $P_0^1$  and  $M_0^1$ . In general, for  $j \leq i$  the node  $X_j^i$  is represented by  $M_{j-1}^i$  and  $P_{j-1}^i$ . Apart from this change, every node in the graph has a corresponding tripartition, and the edges correspond to the relation among tripartitions.  $\square$

While the graphs that can be realized by using this lemma have the quite specific form of  $n$  disconnected subgraphs of  $n$  nodes each, they are sufficient for showing some interesting properties. For example, that  $\Rightarrow$  is neither symmetric nor transitive even when restricting to weakly stable partitions only is proved by a simple graph like  $X \Rightarrow Y$  and  $Y \Rightarrow Z$ ; since it is composed of three nodes, it has to be replicated three times.



Three subgraphs of a simple cycle each are the proof that some sources have no mildly stable partition according to the first definition. Indeed, a set of sources can be realized so that it has exactly these weakly stable partitions; for none it holds that  $X \Rightarrow Y$  implies  $Y \Rightarrow X$ ; therefore, these sources have no mildly stable partition according to the first definition.



This lemma allows for proving that the three definitions of mild stability are different from each other. A graph suffices, since it can be implemented by replicating it for a number of times. Since the lemma only concerns weakly stable partitions, the loops are omitted.

$$\begin{array}{c} X \Leftrightarrow Y \Rightarrow Z \Rightarrow W \\ \Leftarrow \end{array}$$

The relation is defined by:  $X \Rightarrow Y$ ,  $Y \Rightarrow X$ ,  $Y \Rightarrow Z$ ,  $Z \Rightarrow X$  and  $Z \Rightarrow W$ . Since it has four nodes, it has to be replicated in four copies to allow for the application of the lemma. Only the first copy is considered, since the others are identical.

The tripartition  $X$  is mildly stable according to the first definition, since  $X \Rightarrow Y$  and  $Y \Rightarrow X$ . The tripartition  $Y$  is not, since  $Y \Rightarrow Z$  but not the converse. However, since  $Z \Rightarrow X$  and  $X \Rightarrow Y$ , it follows  $Z \xRightarrow{*} Y$ . Therefore  $Y$  is mildly stable according to the second definition but not to the first. Apart from  $W$ , none of the tripartitions is mildly stable according to the third definition since, for example,  $Y \xRightarrow{*} W$  but not the other way around.

This counterexample shows how mildly stable partitions can be visualized: the first definition requires all outgoing edges to form cycles of length at most two; the second is the same without constraints on the length; the third can be recast in terms of maximality.

**Theorem 18** *The mildly stable tripartitions of a set of sources according to the third definition are the maximal tripartitions with respect to the relation  $\xRightarrow{*}$ .*

*Proof.* By definition, a tripartition  $X$  is not maximal if  $X \xRightarrow{*} Y$  for some  $Y$ , but not the converse. This is exactly the definition of  $X$  being not maximal with respect to  $X \xRightarrow{*} Y$ .  $\square$

This theorem implies that every set of sources has some mildly stable partition according to the second definition, since  $\Rightarrow^*$  is by construction reflexive and transitive, and the number of tripartitions is finite. The same can be proved for the second definition.

**Theorem 19** *Every set of sources has mildly stable partitions according to the second definition.*

*Proof.* The relation  $\Rightarrow$  is serial: every tripartition  $X$  has at least a maxset, and this maxset induces a tripartition  $Y$ ; this is the definition of a serial relation: for every  $X$  there exists an  $Y$  such that  $X \Rightarrow Y$ . Visualizing  $\Rightarrow$  as a graph, every node has at least an outgoing edge.

Some tripartitions may not be mildly stable. This is the case if  $X \Rightarrow Y$  holds but  $Y \Rightarrow^* X$  does not. In both cases,  $X$  and all other tripartitions  $Z$  such that  $Z \Rightarrow^* X$  can be removed from consideration. By assumption,  $Y \Rightarrow^* Z$  does not hold; such a removal does not change the tripartitions that are reachable from  $Y$ .

The same argument can be repeated for  $Y$ : either  $Y$  is mildly stable or some tripartitions can be removed from consideration. Since the number of tripartitions is finite, at some point either a mildly stable partition is found or only a single tripartition  $W$  is left. In such a case,  $W \Rightarrow W$  and the partition is strongly, and therefore mildly, stable.  $\square$

## 7 Multiple classes

Dividing the sources in two classes is based on a very clear-cut distinction: a source is either reliable (it always provide right information) and unreliable (may provide wrong information). Also the tripartitions are sharply defined: a source always, sometimes or never provide right information. Partitions in four or more classes cannot be based on a similarly uncontroversial division. Yet, a source providing three correct formulae out of four is more trustable than a source providing one out of five. The problem is where to place the lines.

Even more general mechanisms can be employed. An high-level summary of what is done in the previous sections gives at least two possible directions of generalization. For both bipartitions and tripartitions, the principle is: a. every partition has a set of maxsets; and b. every maxset induces a partition. Abstracting from the specific realization, a partition is a way for representing the relative reliability of the sources, and a maxset is a way of merging formulae given the reliability of their sources. Therefore, the principle is:

1. from a reliability assessment of the sources, define how they can be merged;
2. from a possible result of merging, assess the reliability of the sources.

The reliability of the sources is not necessarily a partition; it can for example be an ordering of the sources, or a numerical evaluation over them. A possible result of merging could be a maxset, but some integration methods output an ordering over the interpretations. Various possibilities are explored in this section.

### 7.1 Percentage bounds

The most obvious extension to the method is to divide the sources on the percentage of right formulae they provide. This allows for as many classes as wanted. For example, the first class contains the sources that provided only correct formulae, the second those providing between 90% and 100% of correct formulae, etc.

A problem with *every* method of this kind is that the bounds are arbitrary: why not setting the division at 89% instead of 90%? Why should a source providing 89% of correct formulae be in the third class while another just a little better at 90% is in the second? This is considered in a further generalization, for the moment the reader is asked to accept a division based on some fixed bounds with some intuitive appeal: the reliability bound is fixed at 90%, unreliability at 50%. These numbers consider a source reliable if it provides 90% or more of correct formulae, unreliable if it provides less than 50% of them. The technical results in this section hold for every other fixed percentages.

Let  $C(S_i, M)$  be the fraction of formulae in the set  $S_i$  that are consistent with the set  $M$ . This function is used with a source  $S_i$  and a maxset  $M$ . Therefore,  $C(S_i, M)$  indicates how reliable the source  $S_i$  is assuming that the state of the word is described by the maxset  $M$ .

$$C(S_i, M) = \frac{|\{A \in S_i \mid M \not\models \neg A\}|}{|S_i|}$$

This function allows for evaluating a source given a maxset  $M$ . In particular, for the two bounds 90% and 50%, it produces the partition of the sources ( $R|P|U$ ):

$$\begin{aligned} R &= \{S_i \mid 0.9 \leq C(S_i, M)\} \\ P &= \{S_i \mid 0.5 \leq C(S_i, M) < 0.9\} \\ U &= \{S_i \mid C(S_i, M) < 0.5\} \end{aligned}$$

Some limit cases are of interest. Setting both boundaries at 0% makes  $P$  and  $U$  empty and  $R$  contain all sources. The only induced partition is that containing all sources in its first class; this partition is strongly stable; all plain maxsets are therefore maxsets of a strongly stable partition.

Fixing the boundaries at 100% and 0% makes  $U$  empty;  $R$  and  $P$  respectively contain the reliable and unreliable sources as in a bipartitions. The same holds if both boundaries are 100%, but this time  $P$  is empty and  $U$  contains the unreliable sources. As in bipartitions, these are the sources that provide from zero to all but one correct formulae. The examples on bipartitions carry to this case. One such examples proves the existence of weakly stable partitions:

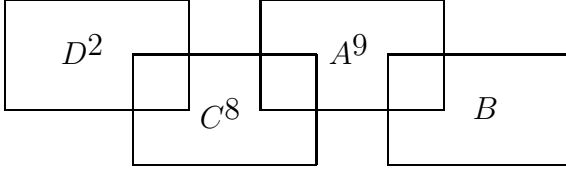
for some bounds, strongly stable partitions exists for all set of sources.

The question is whether strongly stable partitions always exist for an arbitrary pair of percentages. Actually, even the existence of weakly stable partition is no longer obvious. Indeed, even if a maxset induces a partition, it may not be a maxset of it. This was the case for the definitions used in the previous sections, and many proofs rely on this fact.

Given that reliability does not depend only on the presence or absence of formulae consistent with a maxset but also on how many these are, the number of times the same formula is provided by a source matter. To simplify the notation, a formula  $A$  provided nine times by a source is written  $A^9$ . The implicit assumption that sources are multisets instead of sets is not really necessary, as  $A^9$  is essentially equivalent to  $A \vee T_1, \dots, A \vee T_9$ , where each  $T_i$  is a formula with a single model that does not satisfy any other formula; this may require new variables, forgetting which [14] results in nine copies of  $A$ .

**Theorem 20** *For some percentage bounds, the partition induced by a maxset does not have that maxset.*

*Proof.* The claim is proved by a counterexample: the bounds are 90% and 50%, the sources are  $\{A^9, C\}$  and  $\{C^8, D^2\}$ , the plain maxsets are  $\{A, B\}$ ,  $\{A, C\}$ , and  $\{C, D\}$ .



The sources have the following percentage of formulae consistent with the plain maxset  $\{A, C\}$ :

- the maxset is consistent with  $A$  but not with  $B$ ; the percentage of consistent formulae of  $\{A, B\}$  is therefore 90%;
- the maxset is consistent with  $C$  but not with  $D$ , 80% of  $\{C, D\}$ .

The induced partition is therefore  $((A^9B)|(C^8D^2))$ . Since  $\{A, B\}$  is the only plain maxset that is consistent with all formulae in the first class of the partition, it is the only maxset of the partition. Therefore,  $\{A, C\}$  induces this partition but is not a maxset of it.  $\square$

The proof that every set of sources has a weakly stable partition was based on the property that the partition induced by a maxset has that maxset. It therefore no longer works for percentage bounds. The claim still holds, but for a different reason, suggested by the above proof: a plain maxset is not a maxset of the induced partition because of another maxset that is “more consistent” with the formulae in the first class of the partition. In turn, this maxset may not induce a weakly stable partition for the same reason, but at some point a maximum is reached.

**Lemma 12** *For every pair of bounds there exists an irreflexive, antisymmetric and transitive ordering  $\prec$  among plain maxsets such that, if  $M$  is not a maxset of its induced partition then  $M \prec M'$  for some maxset  $M'$  of this partition.*

*Proof.* The bounds define how a maxset  $M$  induces a partition  $(R|P|U)$ . In the following proof the percentage bounds are set at 90% and 50% for the sake of explanation, but every other pair would work. The ordering is based on the following function.

$$e(M) = (M \cap R, M \cap P) \text{ where } (R|P|U) \text{ is the partition induced by } M$$

The function  $e(M)$  does not have the partition as an argument because the partition is the one induced by  $M$ . For example,  $e(M')$  is defined in terms of the partition  $(R'|P'|U')$  induced by  $M'$ .

Let  $(X, Y) \prec (Z, W)$  if either  $X \subset Z$  or  $X = Z$  and  $Y \subset W$ . The ordering on maxsets is defined by  $e(M) \prec e(M')$ . This ordering is by construction irreflexive, symmetric and transitive. Remains to prove that if  $M$  is not a maxset of its induced partition then  $e(M) \prec e(M')$  for some maxset  $M'$ .

Let  $M$  be a plain maxset that is not a maxset of its induced partition  $(R|P|U)$ . Since  $M$  is consistent but not a maxset of  $(R|P|U)$ , another consistent set of formulae is strictly larger than  $M$  within  $R$  or within  $R \cup P$ . Such a set can be added some formulae to obtain a maxset  $M'$  with the same property: either  $M \cap R \subset M' \cap R$  or  $M \cap (R \cup P) \subset M' \cap (R \cup P)$ . These conditions can be rewritten as:

1.  $M \cap R \subset M' \cap R$ ; or

2.  $M \cap R = M' \cap R$  and  $M \cap P \subset M' \cap P$ .

Let  $(R'|P'|U')$  be the partition induced by  $M'$ . In both Case 1 and Case 2,  $M \cap R \subseteq M' \cap R$ . By definition of induced partition, if  $S_i \in R$  the maxset  $M$  contains at least 90% of the formulae in  $S_i$ . Since  $M \cap R \subseteq M' \cap R$ , also  $M'$  contains at least 90% of  $S_i$ , implying that  $S_i \in R'$ . This holds for every source in  $R$ ; therefore,  $R \subseteq R'$ .

In the first of the two cases above,  $M \cap R \subset M' \cap R$ . Since  $R \subseteq R'$  it follows that  $M \cap R \subset M' \cap R'$ . Therefore,  $e(M) \prec e(M')$ .

In the second case,  $M \cap R = M' \cap R$  and  $M \cap P \subset M' \cap P$ . Since  $R \subseteq R'$ , the equality implies  $M \cap R \subseteq M' \cap R'$ , which is the first condition for  $e(M) \prec e(M')$  to hold.

If a source is in  $P$ , then  $M$  contains between 50% and 90% of its formulae. Since  $M \cap P \subset M' \cap P$ , then all these formulae are also in  $M'$ . But  $M'$  may also contain other formulae of this source; therefore, this source can be in  $P'$  but also in  $R'$ . Formally,  $P \subseteq P' \cup R'$ .

Let  $F$  be a formula in  $M' \cap P$  that is not in  $M \cap P$ , and  $S_i$  the source containing it. Since  $F$  is in  $P$  but not in  $M \cap P$ , it is not in  $M$ . From  $S_i \in P$ , it follows  $S_i \in R' \cup P'$ . If  $S_i \in R'$  then  $M' \cap R'$  contains  $F$  while  $M \cap R$  does not (because  $F \notin M$ ). This alone implies  $e(M) \prec e(M')$ . Otherwise,  $F$  is in  $P'$ . Therefore,  $M' \cap P'$  contains  $F$  while  $M \cap P$  does not, which is the second condition for  $e(M) \prec e(M')$  to hold.  $\square$

This lemma is in a way similar to Lemma 5 for bipartitions, in that it relates a maxset with a maxset of its induced partition. Differently than that it establishes such a relation only if the first maxset is not a maxset of its induced partition, and only with a single maxset of the induced partition. It is however sufficient to prove the existence of weakly stable partitions.

**Theorem 21** *For every pair of bounds, weakly stable partitions exist for every set of sources.*

*Proof.* Let  $M$  be a plain maxset and  $(R|P|U)$  its induced partition. If this partition has  $M$  as a maxset, it is weakly stable and the claim is proved. Otherwise, by the previous lemma there exists  $M'$  such that  $e(M) \prec e(M')$ . The same argument can be repeated for  $M'$ : either its induced partition has  $M'$  as a maxset, or another maxset  $M''$  exists such that  $e(M') \prec e(M'')$ . Since this relation is irreflexive, antisymmetric and transitive and the set of maxsets is finite, at some point a maxset  $M^*$  is reached such that  $e(M^*) \prec e(M''')$  does not hold for any other maxset  $M'''$ , proving that  $M^*$  is a maxset of its induced partition.  $\square$

## 7.2 Ordering

A different way to implement the principle of evaluating the reliability of the sources from a possible result of merging is to use orderings for both. In particular, the relative reliability of sources can be encoded as an ordering over sources, and a possible result of merging as a priority order over the propositional interpretations. Such representation requires a priority order over interpretations to allow evaluating the reliability of the sources, and an evaluation of the reliability of the sources to induce one or more priority orderings over the interpretations.

The reliability of sources can be expressed by a relation  $\leq$  and the priority over interpretations by a relation  $\preceq$ . When restricted to total orders, these can be recast in terms of functions from sources and interpretations to integers. The reliability of a source  $S_i$  is denoted  $\text{reliability}(S_i)$ , the priority of an interpretation  $I$  is denoted  $\text{priority}(I)$ . They are related by:

**reliability**  $\implies$  **priority**:  $\text{priority}(I) = \min\{\text{reliability}(S_i) \mid I \models A \text{ and } A \in S_i\}$

**priority**  $\implies$  **reliability**:  $\text{reliability}(S_i) = \max\{\text{priority}(A) \mid A \in S_i\}$  where  $\text{priority}(A) = \min\{\text{priority}(I) \mid I \in A\}$ .

If an interpretation  $I$  is satisfied by a formula in a source of reliability 1 and one in a source of reliability 2, then  $I$  has priority 1. In general, an interpretation is likely being the actual state of the word if a reliable formula supports it. This explains the minimization.

On the converse, a source providing a formula that is considered untruthful is assessed as unreliable. The evaluation of both sources and interpretations is qualitative: the number of formulae supporting an interpretation does not matter, as well as how many formulae a source provides.

When comparing two reliability orderings expressed in numeric form, normalization is necessary. Indeed, the orderings in which  $I$  is evaluated 0 in both while  $J$  is respectively evaluated 1 and 2 may be the same if no model is evaluated to 1 in the second. In other words, removing the “empty levels” is necessary when comparing two orderings.

An example is shown on the following sources:

$$\begin{aligned} S_1 &= \{x, y\} \\ S_2 &= \{\neg x \wedge y\} \end{aligned}$$

A stable ordering is  $\text{reliability}(S_1) = \text{reliability}(S_2) = 0$ . All models of either of the three formulae have priority 0, the other model  $\{\neg x, \neg y\}$  has priority 1. As a result, all three formulae have reliability 0, and therefore both sources have reliability 0 as well. This is the original reliability ordering, which is therefore stable.

Not all orderings are stable, as shown by  $\text{reliability}(S_1) = 0$  and  $\text{reliability}(S_2) = 1$ . The priority of models is  $\text{priority}(\{x, y\}) = \text{priority}(\{x, \neg y\}) = \text{priority}(\{\neg x, y\}) = 0$ , which leads to  $\text{reliability}(S_2) = 0$ .

### 7.3 Weighted Merge

The implicit assumption when using maxsets is that merging consists in collecting as many formulae as possible while retaining consistency. The result is a subset of the formulae provided by the sources. Semantically, a model of the merged formulae can only be one satisfying at least one of the formulae; typically it satisfies more because of the maximality, but at least one has to be satisfied.

A different approach [12, 10, 11] is based on distances, so that a model may be considered even if it does not satisfy any formula, but is sufficiently close to all of them.

A number of different definitions have been provided, but often the distance  $d(I, F)$  of a model  $I$  to a formula  $F$  is the minimal Hamming distance from  $I$  and a model  $J$  of  $F$ . Such a model  $I$  is good if the distance is low, and bad otherwise. The distance is calculated for any of the given formulae  $F_1, \dots, F_m$ ; the badness of  $I$  is a combination of them, for example a weighted sum.

$$d(I, F_1, \dots, F_m) = w_1 \times d(I, F_1) + \dots + w_m \times d(I, F_m)$$

The models at minimal badness form the result of merging. This minimization can be applied to all models or only those satisfying some integer constraints. The assumption is



that usually some facts are known for certainty, so that a number of models can be ruled out simply because they conflict with them.

The role the partitions had in the previous sections is now taken by the weights. They both formalize the reliability of the sources, and therefore the measure of how much they have to be taken seriously when merging. In the same way, the possible ways of merging were the maxsets then and are the models now. The same mechanism of obtaining a possible result of merging from a reliability assessment, and then to compute a reliability assessment from it, can therefore be applied.

Let  $S_1, \dots, S_m$  be a set of sources, each comprising one or more formulae. Each source  $S_i$  has a weight  $w_i$ . When merging, all formulae in a source  $S_i$  have weight  $w_i$ . This is the same principle of merging with maxsets: the reliability of a source is taken as a whole, and then every formula it provides is assumed to have that reliability.

Given the weights, the models at minimal combined distance are defined. This set of models is the result of merging. If this result is correct, the distance between it and the formulae in a source allows evaluating the reliability of the source. Using a model at time or all of them at once allows for obtaining the definitions of weak and strong stability. The first is defined as follows.

- merging sources  $S_1, \dots, S_m$  with weights  $w_1, \dots, w_m$  produces the models at minimal combined weighted distance from the formulae in the sources;
- the reliability of a source  $S_i$  from a model  $I$  is the maximum distance between  $I$  and a formula in  $S_i$ .

The reason for the asymmetry between the two definitions is that the reliability of a source is not its most truthful formula but its least: a source is not trustable if it provides some formulae that are very far from the truth, even if it also provides some that are close. Other way of combining the distances can be used, such as the average.

The principle of operation is to start with a set of weights, compute the possible results of merging, and from these compute the distance of the sources. These should be the same as the original ones, at least qualitatively: if a source has a weight greater than another, then its resulting distance should be lower than that of the other.

The following example shows that some weights may be weakly stable while others are not. This means that the fixpoint definition allows excluding the second set, and therefore restricting the possible choices.

$$\begin{aligned}
S_1 &= \{A, B\} \\
S_2 &= \{C\} \\
A &= \neg x_1 \wedge \neg x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \\
B &= x_1 \wedge x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_6 \wedge \neg x_7 \\
C &= x_1 \wedge x_2 \wedge x_3 \wedge x_4 \wedge x_5 \wedge x_6 \wedge x_7 \\
K &= x_1 \wedge x_2 \wedge \neg x_3 \wedge \neg x_4 \wedge \neg x_5 \wedge \neg x_7 \\
I &= \{x_1, x_2, x_3, x_4, x_5, \neg x_6, \neg x_7\} \\
J &= \{x_1, x_2, x_3, x_4, x_5, x_6, \neg x_7\}
\end{aligned}$$

The formula  $K$  expresses the constraints: only its two models  $I$  and  $J$  are considered when computing the result of merging. The distance between these models and the formulae are:

$$\begin{aligned}
d(I, A) &= 4 \\
d(I, B) &= 2 \\
d(I, C) &= 2 \\
d(J, A) &= 5 \\
d(J, B) &= 3 \\
d(J, C) &= 1
\end{aligned}$$

The starting point of the method is to assume some weights for the sources, for example  $w_1 = w_2 = 1$ ; this is an obvious choice: in lack of information indicating which source to trust more, they are trusted the same.

Using these weights, the badness of  $I$  is  $w_1 \times (d(I, A) + d(I, B)) + w_2 \times d(I, C) = 1 \times (4 + 2) + 1 \times 2 = 8$ . The badness of  $J$  is  $1 \times (5 + 3) + 1 \times 1 = 9$ . The result of merging is therefore  $I$  only. This allows for an easy evaluation of the sources: 4 for  $S_1$  and 3 for  $S_2$ . These reliability measures are different, contradicting  $w_1 = w_2$ . The original choice of weights is not stable.

A different outcome is obtained by  $w_1 = 1$  and  $w_2 = 2$ . The badness of  $I$  is  $1 \times (4 + 2) + 2 \times 2 = 10$  and that of  $J$  is  $1 \times (5 + 3) + 2 \times 1 = 8 + 2 = 10$ . Both models are in the result of merging, so both have to be considered when checking stability. The distance between  $I$  and  $S_1$  is 4, between  $I$  and  $S_2$  is 2. These compare the opposite of  $w_1$  and  $w_2$ , as required (large weight means high reliability, which in turns should produce a lower distance). This proves that the weights are weakly stable. A similar calculation for  $J$  shows that the same happens for it, making the weights strongly stable.

## 8 Directions

The principle studied in this article is that some reliability orderings can be excluded on the ground that using them for revising leads to a different ordering. The way this principle is implemented is by assuming sources to generate a single or a set of formulae, that the reliability ordering is represented by a partition of the sources or by weights, and that merging is done by maxsets or by a weighted sum.

Apart from how the reliability ordering and the merging process are realized, the principle itself can be extended in many ways.

### 8.1 Sure information

This is possibly the direction of most interest: some sources of unknown reliability provide formulae to be merged, but some information is also known for certain. This is a common scenario in belief merge, encoded by integrity constraints [13] as follows: only the models that satisfy them are considered when selecting the ones that make the result of merging.

The fixpoint definition allows for a smooth encoding of formulae that are considered certain. They are collected in a single source, and only reliability orderings having that source is in the first class are considered. Since the first class of an induced partition contains only formulae that are consistent with the result of the merge, this makes such result to be consistent with all these formulae.

Formalizing the certain information as a source has the advantage of an uniform treatment of it along with the other formulae. There is no prior reason why another source could not

be equally certain; if so, it is in the first class of the reliability partition, together with the source encoding the sure formulae. In fact, the integrity constraints come from somewhere, the only difference with the other formulae being that their sources can never be considered unreliable, not even partially so.

The reason why this variant may be the case of most interest stems from having certain information to rule out some reliability orderings. The scheme with no integrity constraint uses only the regular sources to reckon their reliability ordering. The integrity constraints ground this evaluation on information that is known for certain.

## 8.2 Partial reliability ordering

Limiting to reliability orderings with a given source in the first class is only a particular case of partial information over the reliability ordering. If  $S_0$  is such a source,  $S_0 \leq S_i$  holds for every other source  $S_i$ .

More generally, an arbitrary partial knowledge of the ordering can be assumed. For example, it may be that only  $S_1 < S_2$  and  $S_2 \leq S_3$  are known. This rules out the partition  $(S_2|S_1S_3)$  but neither  $(S_1|S_2S_3)$  nor  $(S_1|S_2|S_3)$ . Since bipartitions and tripartitions set a strong division between classes (all true, some true, none true), constraints of type  $\leq$  are better suited for them, as otherwise an apparently harmless statement like “ $S_1$  is strictly more reliable than  $S_2$ ” would turn “ $S_1$  is only partly reliable” into the strong implication that  $S_2$  is totally unreliable. Such strict constraints like  $S_1 < S_2$  are better left to multiple-classes partitions, where they only state that  $S_2$  provides fewer correct formulae than  $S_1$ .

## 8.3 Untrustable sources

Tripartitions class sources in three groups: totally reliable, partly reliable and unreliable. The first two are clearly defined, as they are the sources that provide only correct formulae and some correct formulae. However, the meaning of the third class changes depending on the considered direction:

**partition**  $\rightarrow$  **maxset** formulae in the third class are included if consistent with the formulae of the first two;

**maxset**  $\rightarrow$  **partition** a source goes in the third class if all its formulae are inconsistent with the maxset.

In the first point formulae are included only if possible, in the second they are not. Only if the partition is stable the two coincide. This highlights the ambiguity of defining a source “totally unreliable”. It may be a source that is less reliable than all others but in principle may provide correct information, but it may also be a source that always provide wrong information.

The third class can therefore be split in two. A maxset may include formulae in the first subclass if consistent with the rest of it, but does not include any of the second, simply because these are known being false. In a way, this fourth class is specular to the first: all its formulae contradict the truth rather than being consistent with it. Such a class can formalize scenarios in which a source intentionally tries to lead the revision process in a certain direction; this raises the question of detecting strategies such as providing some right information to escape the classification in the fourth class.

## 8.4 Flexible bounds

Dividing the sources on 100% and 0% of correct formulae, as in bipartition and tripartitions, appeals to a qualitative notion of reliability and unreliability, every other bound (like 80% or 50%) is difficult to motivate over others (like 90% and 51%).

A possible solution is to assume that a partition like  $(S_1|S_2S_3|S_4)$  only means that there exists a reason to assume that  $S_1$  is more reliable than both  $S_2$  and  $S_3$ , which are both more reliable than  $S_4$ , without requiring two fixed borders between them. As an example, this partition is acceptable if the four sources provide 80%, 72%, 69% and 30% correct formulae. The bound can be fixed at 75% and 50% to support this division, but also at 79% and 60%, or at 79% and 40%.

This mechanism lifts the decision of an exact pair of percentages of correct formulae that define a source reliable or partly reliable. A partition like  $(S_1|S_2S_3|S_4)$  is then stable if such a pair makes the partition itself induced from the result of merging.

## 8.5 Technical results

Some other possible expansions of this work regards the study of the existing definitions, rather than the extensions of them. Lemma 11 allows for the creation of a scenario with given partitions and maxsets, but only if these satisfy certain conditions. It would be interesting to find out whether these constraints could be lifted.

The two algorithms presented in this article cover the case of bipartitions only. Extending them to tripartitions or even to the case of arbitrary partitions would make them applicable to more general situations. Also, the complexity of inferring from the maxsets of the stable partitions, or checking for the existence of strongly stable tripartitions, is also an open problem.

## 9 Conclusions

The principle of excluding some reliability orderings when the merging result they produce conflicts with them can be implemented in many ways. In the simplest case, sources are classified as reliable and unreliable. The second group can be split into the partly reliable and unreliable sources.

The change has a big impact on the technical results: while strongly stable bipartitions always exist, this is not the case for strongly stable tripartitions. This motivates the definition of an ordering among weakly stable partitions in order to obtain a form of stability that is more grounded than weak stability.

Switching to classes based on percentages makes another important property fail: the partition induced by a maxset may not have that maxset. The existence of weakly stable partitions for arbitrary pairs of bounds, which was a simple consequence of this fact, had to be proved by a different argument.

All of this indicates that the technical consequences are strongly dependent on the details of the definition employed. While the general principle is always the same (fix a reliability ordering, merge using it, check if the result agrees with the reliability ordering), the properties of the framework depend on how it is implemented.

However, some general facts appear to hold regardless:

1. weakly stable partitions always exist; this is not obvious from the definition, rather the opposite: a definition of partitions and merge that does have this property is unusable

since weak stability is the most basic requirement for merging; if some sources do not have such a partition, they cannot be merged; it is hard to imagine a scenario where information cannot be integrated, not even with an void result;

2. since maxsets induce partitions and partitions have maxsets, an ordering of the partitions (or maxsets, or both) always exists; such an ordering can be used to define a form of maximality of partitions, or at least to exclude partitions that are strictly dominated by others; the rationale is that if a reliability ordering lead to a result of merging that implies a different reliability ordering, the latter is more grounded than the former;
3. as in the general problem of belief revision, the aim is at the same time to obtain some merging result but also to restrict the possible alternatives as much as possible, since this leads to a resulting formula that carries as much information as possible; as a result, whichever pair of definitions are used for the result of merging and the induced reliability ordering, some stable partitions should always exists, but as few as possible; this is why definitions that lead to cases without weakly stable partitions are ruled out; at the same time, an inflation of weakly stable partitions (like when the lead to all plain maxsets) motivates the introduction of strong stability or at least the restriction to maximal weakly stable partitions.

Few articles in the literature are related to the research presented in this article. This is because most work in belief revision and merging is about going from a set of formulae to their aggregation. A handful of articles consider the reverse problem: Haret, Mailly and Woltran [8] divide a formula into simpler formulae that give the original one when merged; Booth and Nittka [3] and Liberatore [15] use the history of previous revision to obtain reliability information; Liberatore [16] use merging samples to the same aim. All this research share the reversal of the traditional role of the sources and the result of merge, where the first are input and the second is output. Apart from the first cited article, which aims at representing knowledge in some simple form, the other do this to obtain reliability information, which can be then used in the following merging. The present article does the same, but does not assume any prior knowledge. Rather, the role of history or examples is taken by the merging result itself, which allows to evaluate the sensibility of the reliability ordering initially assumed.

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